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Name. Surendra Sharma

Sub. Computational
Technique (CT)

Std. BCE IV Year

Div. 078 BCE 6H

Roll No. 078 BCE 178

Teacher's

Surendra Sharma

FEM

• Why study?

↳ ~~become~~ an integral part of design and development of numerous engineering systems.

→ Nature operates via gradients

→ Most eng. systems are mathematically represented via gradients ($\frac{df}{dx}$, $\frac{d^2f}{dx^2}$, ∇f) or the differential equations.



Cauchy's equation in solid Mechanics

Navier-Stokes eqs in Fluid Mechanics

Maxwell's eqs in Electromagnetics

} All have differential equations.

→ In eng. problems, along with differential equations, we also have additional constraints called Boundary conditions (BCs).

→ Differential equations + BCs → Boundary Value Problem (BVP)
(defined for a specific domain or geometry)

→ Solutions to BVP

② ↳ Analytical methods (Direct integration, Laplace transforms, etc.) are applicable only for simple geometries.

↳ Handling complex geometries? (Real world)

↳ Using Numerical Methods.



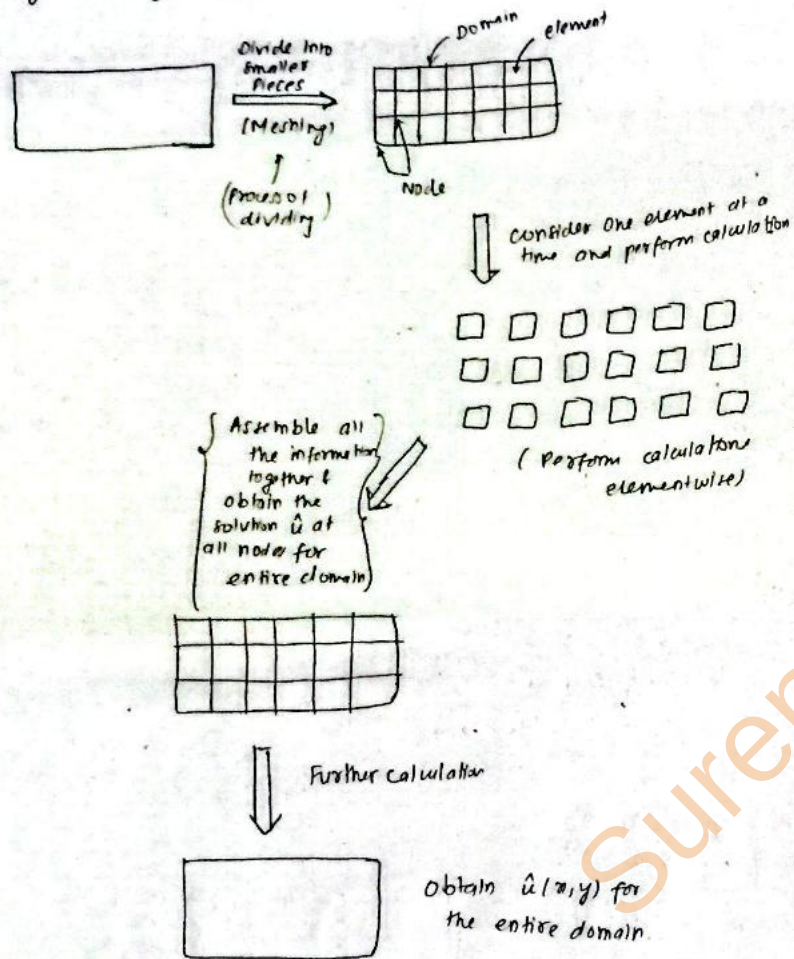
is a numerical method used to obtain an approximate solution for a given BVP.



↳ Linear system of equations are solved using a computer (Gauss elimination algorithm)

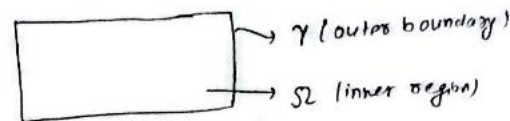
Process of FEM

- say, objective is to find the (approx) solⁿ $\hat{u}(x,y)$ to the boundary value problem.
- Taking rectangular plate (but applicable for any shape)



∞ Divide and conquer

Mathematical treatment



BVP

- Poisson's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \phi$$

(+ Boundary conditions)

- If $u^e(x,y)$ is an exact solution, then

$$\Rightarrow \frac{\partial^2 u^e}{\partial x^2} + \frac{\partial^2 u^e}{\partial y^2} - \phi = 0$$

- Let $\hat{u}$ is an approx solution:

$$\Rightarrow \frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} - \phi \neq 0 \quad (\text{not exactly satisfied})$$

$$\Rightarrow \text{Residual } R = \frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} - \phi$$

- Now, objective is to find an approximate solution $\hat{u}$ such that R is close to zero at each point within our domain Ω

- Weighted-Residual (WR) statement

$$\rightarrow \int \omega(x,y) \left(\frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} - \phi \right) d\Omega = 0 \quad (10)$$

$$\rightarrow \int \omega(x,y) (\nabla^2 \hat{u} - \phi) d\Omega = 0 \quad \left\{ \omega(x,y) \rightarrow \text{known weighted function} \right\}$$

- Weak form of the WR statement:

$$\rightarrow \int \omega \nabla \hat{u} \cdot \hat{n} d\Gamma - \int \omega \nabla \hat{u} \cdot \nabla \hat{u} d\Omega - \int \omega \phi d\Omega = 0$$

(Defined for entire domain)

∞ After Meshing $\int \omega \hat{u} \cdot \hat{n} d\Gamma_i - \int \omega \nabla \hat{u} \cdot \nabla \hat{u} d\Omega_i - \int \omega \phi d\Omega_i = 0$

(Defined for each element) (ith element)

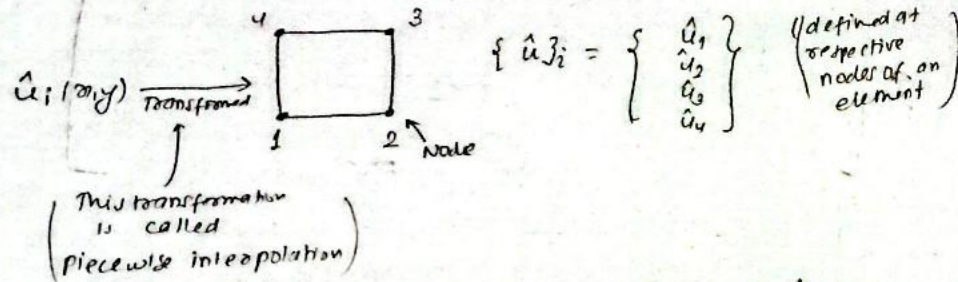
- Let's isolate any element

$$\square \hat{u}_i(x, y)$$

Any element is a continuous domain and a collection of infinite points

$\hat{u}_i(x, y) \leftarrow$ field quantity or function defined for element i (complicated)

will be simpler if we had,



$$\hat{u}(x, y) = N_1(x, y)\hat{u}_1 + N_2(x, y)\hat{u}_2 + N_3(x, y)\hat{u}_3 + N_4(x, y)\hat{u}_4$$

OR

$$\hat{u}(x, y) = \sum N_k u_k \text{ or } [N]\{u\}$$

$N_1, N_2, N_3, N_4 \rightarrow$ shape functions in FEM (usually known)
 $\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4 \rightarrow$ unknown numbers

Substitute

$$\int \omega \nabla \hat{u} \cdot \hat{n} d\gamma - \int \nabla \omega \cdot \nabla \hat{u} d\Omega - \int \omega f d\Omega = 0$$

Weak statement

$$[K]_i \{u\}_i = \{f\}_i \quad (\text{linear system of equation})$$

↑ element stiffness matrix ↑ element load or force vector

- Adding together the contribution of all elements we obtain, linear system of equation for whole domain

$$[K]\{u\} = \{F\}$$

↑ Global stiffness matrix ↑ Global Force vector

(No. of eqs = No. of nodes)

Summary

- Start with BVP
- Write weighted residual statement
- Obtain the weak form
- Use interpolation equation for element quantities
- Assemble to obtain global quantities

Power of FEM

Certification requirement for car (crash testing analysis) → to keep passengers safe

Assume • Analyst requires 1000 cars (full scale testing)
 (costly & time consuming)

• Alternatively, we crashed 100 cars - physical testing
 800 cars - FEM simulations (computer)

∴ FEM enables virtual testing platform that saves cost & time.

(P. Seshu) → FEM

FEM (efficient eng. m)

powerful numerical technique that uses computational power to calculate approximate solutions to complex geometry, loads or material (real world problems)

solid mechanics: static analysis; dynamic analysis; Buckling analysis; Modal analysis

Fluid Mechanics

Heat transfer

Electromagnetics

FEM solves the complex problem by splitting the body into a number of smaller elements that are connected together at nodes (Discretization)

Collection of nodes & elements $\rightarrow$ Mesh

Several different element shapes used $\rightarrow$ Triangular, Quadrilateral

First order elements $\rightarrow$ (surface elements) $\rightarrow$ 2-D elements typically used to model thin surfaces $\rightarrow$ Δ , $\square$

Solid elements $\rightarrow$ For 3-D bodies $\rightarrow$ Δ , $\square$

Line elements $\rightarrow$ ---

Second-order elements $\rightarrow$ Have additional mid-side nodes and are more accurate



For beam element $\rightarrow$ 3 DOF per node (6 in total)
In 3-D: 6 DOF per node.
Shell element node $\rightarrow$ 3 DOF (in 2-D) (total 12 DOF)

Stiffness

For spring, $F = -kx$
 $\uparrow$ spring stiffness

$\{f\} = [k] \{u\}$
 $\uparrow$ nodal forces $\uparrow$ stiffness matrix of the element (square matrix)
No. of rows = No. of columns = No. of DOF

Stiffness matrix:

- sparse $\rightarrow$ it contains lot of zeros
- Banded $\rightarrow$ the non-zero terms are grouped around the diagonal
- Symmetric $\rightarrow$ for linear-elastic problems
- varies with number of nodes present

Boundary condition $\rightarrow$ known displacements at specific nodes

$$\{F\} = [K] \{U\}$$

Conjugate gradient method.

$\{u\} \xrightarrow{E = \frac{\Delta L}{L_0}} \text{calculate strain } \epsilon \xrightarrow{\sigma = E\epsilon} \text{calculate stress } \sigma$

Stiffness matrix $\rightarrow$ derived by concept of equilibrium
(describe problem in integral form) $\rightarrow$ strong form $\rightarrow$ (i) Direct Method (diff. eq. + BC) $\rightarrow$ for simpler elements
weak form $\rightarrow$ (ii) Variational Method
(iii) Galerkin Method of Weighted Residuals $\rightarrow$ for complex elements

Variational Method

↳ Principle of Minimum potential Energy.

"The displacement configuration that satisfies equilibrium condition is the one that minimises the total potential energy."

$$\Pi = U_e + V$$

$\uparrow$ Total PE $\uparrow$ Strain Energy $\uparrow$ PE of external loads

• Galerkin Method of weighted Residual.

$$\Rightarrow y^*(x) = \sum_{i=1}^n c_i N_i(x)$$

$\uparrow$ Approximate solution $\uparrow$ coefficient $\uparrow$ Trial function

$$\rightarrow y^*(x) = c_1 N_1(x) + c_2 N_2(x) + \dots + c_n N_n(x)$$

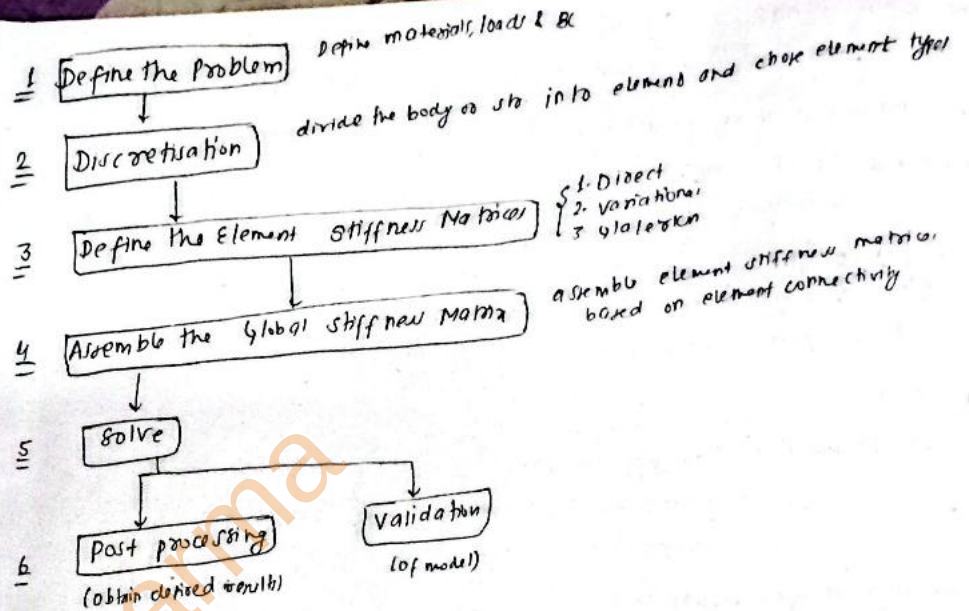
Substituted in diff^l eqⁿ

$$R(x) = D(y^*(x), x)$$

$\uparrow$ Residual
 (is zero if $y^*(x)$ is the exact solution)

$$\rightarrow \int_a^b N_1(x) R(x) dx = 0 \quad \left| \quad \int_a^b N_2(x) R(x) dx = 0 \quad \right| \quad \int_a^b N_n(x) R(x) dx = 0$$

↳ solve for the unknown coefficients.



$\left[\begin{matrix} 3 \\ 4 \\ 5 \end{matrix} \right] \leftarrow$ Software task.
 $\left[\begin{matrix} 1 \\ 2 \\ 6 \end{matrix} \right] \leftarrow$ Engineer task

Some Basic Terms

- ① Domain: Region geometry where our study is considered
- ② BC: known or given condition on some part of the domain
- ③ Initial condition (IC): condition at the starting of the simulation

→ In FEM, a complex region defining a continuum is discretized in simple geometric shapes called finite elements (gives approximate solution)

Applications

- Can be used to analyze structure
 - computation of deflection, stress, strain, force, energy, vibration, etc.
- Non structural problems
 - Heat transfer, fluid flow, seepage, distribution of electric and magnetic potential.

Application software

- ABAQUS, ANSYS, SAP2000, ETABS, STAADpro, etc.
- Research + Design

Basic steps in FE Analysis

- ① * Discretization of continuum into small interconnected element/mesh called finite elements connected at nodes
- ② * Identifying/defining of unknown displacement at nodes
 - * Choice of approximating function.
 - * Formation of element stiffness matrix
 - * " " overall " "
 - * " " element loading matrix
 - * " " overall " "
- ③ * Incorporation of boundary conditions
- * Solution of simultaneous equations.
- * Calculation of stresses or stress resultants.
- * Interpolation of field variable at any pt. of domain using shape functions.

Finite Difference Method (FDM)

- ↳ the solution technique in which differential equations are solved by converting them into difference equations

Applications

- solve problem related to hydraulics, surface water, ground water, hydrology, etc.

Software: HEC-RAS, MikeShe, etc.

Advantages

- ↳ Simple and easy to apply on problems related to hydraulics, hydrology and ground water.

Disadv.

- ↳ difficult to apply for irregular geometry.

Similarities b/w FDM & FEM

- Grid points in FDM are analogous to nodal points in FEM.
- In both, differential equation is converted into algebraic equations

Discrete Element Method (DEM)

→ DEM is a finite difference numerical method for predicting the motion of individual and independently moving objects.

→ Given some particles with their current positions & velocities

↓
Calculate forces
(gravity, resistance from contact with other particles, friction)

↓
Using the eqs of motion, we can determine the future position based on the current velocity.

$$\left(\begin{aligned} x_0 &= at + \frac{1}{2} at^2 \\ v_0 &= v + at \end{aligned} \right)$$

↓
Similarly we can calculate future velocity.
(for small increment in time called time-step)

↓
This process is repeated for each particle in the system for each time step.

→ When a collision occurs b/w particles, particle is treated as a very stiff spring to calculate reaction forces (d friction)
($F = kd$)

→ Add the forces to calculate new position and velocity.

Applications

- Geophysics / Seismology
- Rock Fracture
- Soil Mechanics
- Ice blocks floating into bridge supports
- Industrial / commercial applications.

Discrete Element Method (DEM) or Distinct Element Analysis

→ a numerical method for computing the motion and effect of a large number of small particles

→ Fundamental assumption of this method is material consist of separate, discrete particles of different shapes and properties.

→ The simulation is started by first generating a model, which results in spatially orienting all particles and assigning an initial velocity. The initial data, the relevant laws and contact models are used to compute the force acting on each particle. These forces are added up. Change in position and velocity of particles in next time step due to the force is evaluated by integrating method. Next position are evaluated and the forces again. The iteration is repeated until simulation completes.

Adv.

- extensively used for calculating forces, stress, strain, displacement of a granular material containing large number of particles.
(Can be used in masonry buildings, masonry element interaction)

Disadv.

- Since, the computation of change in position and velocity of each particle needs to be computed during different time steps, it involves a number of loop, hence it is somewhat tedious.

→ granular elements

Boundary Element Method (BEM)

- ↳ a numerical computational method of solving partial differential equations which have been formulated as integral equations (i.e. boundary integral form).

↳ applied in many engineering application including Fluid Mechanics.

✓ The integral equation may be regarded as an exact solution of the governing partial differential equation. The boundary element method attempts to use the BC to fit boundary values into the integral equation, rather than values throughout the space defined by partial differential equation. Once this is done, in the post-processing stage, the integral equation can then be used again to calculate numerically the solution directly at any desired point in the interior of solution domain.

Adv.

- ↳ saves processing time and reduces dimension (discretization of domain only on the boundary)

Disadv.

- ↳ solution obtained may not be as good as FEM, where entire domain is discretized to element.

Application

- ↳ For solution of Laplace, Helmholtz and Maxwell problems

Smooth Particle Hydrodynamics (SPH)

- ↳ computational method used for simulating fluid flow

↳ Mesh-free Lagrangian Method (where the co-ordinates moves with the fluid).

- ↳ works by dividing the fluid into a set of discrete elements, referred as particles. These particles have spatial distance known as smoothing length.

↳ The physical quantity of any particle can be obtained by summing the relevant properties of all the neighbouring particles.

↳ The contributions of each particle to a property are weighted according to their distance from the particle of interest.

Adv.

↳ It can be used in fluid flow analysis.

↳ suitable for deformable boundary problems and moving surface.

Disadv.

↳ Since, it is only related to fluid dynamics, it can't be used in vibration field related to structural analysis like FEM.

History of Numerical computation :-

History of FEM

- Foundation of FEM has been laid independently by scientists, engineers and physicists.
- In 1943, coisont → developed method to use piecewise continuous functions defined over triangular domain.
- In 1950, Polye, Hersh and Weinberger: showed interest in coisont concept.
- In 1956, Turner → developed stiffness matrix for towers, beams and other elements.
- In 1960, Clough → used FEM for first time in plane stress analysis.
- In late 1960s and 1970s, → improvements in the speed and memory capacity of computers contributed to the success in the field of FEM.
- The elastic analysis of the plane stress and plane strain problem has been successfully used in analysis of 3D problems, non-linear, fluid flow and heat transfer.

History of FDM

- ↳ Oldest computational technique.
- Foundation of FDM was laid by Courant, Friedrichs and Lewy in 1928 by publishing Fundamental paper on the solution of problems on mathematical physics.
- After emergence of high-speed computers after 1950, FDM was widely used.

Assignment -1

Compute program

↳ ask two input matrix (A and B) of size $n \times n$

↳ 1. Addition of two matrix (A+B)

↳ 2. Product (AxB and BxA)

↳ 3. Add. const. term 5 to each element of matrix A and add it to 5x times of matrix B

Ch-2

Solving techniques

(1) Cramer's method.

$$x_1 = \frac{\Delta_1}{\Delta}$$

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} d_1 & a_2 & a_3 \\ d_2 & b_2 & b_3 \\ d_3 & c_2 & c_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & a_3 \\ b_1 & d_2 & b_3 \\ c_1 & d_3 & c_3 \end{vmatrix}$$

(2) Gauss elimination:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

(only row operations)

- Make upper or lower triangle zero.
- FW substitution or back substitution.

(3) Gauss Jordan

$$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & d_1 \\ b_1 & b_2 & b_3 & d_2 \\ c_1 & c_2 & c_3 & d_3 \end{array} \right]$$

both row & column operations

Convert to identity matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array} \right]$$

on converting to identity matrix, these give the values of variable

(*) Gauss seidal method (Iterative method)

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= c_2 \\ \vdots & \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= c_n \end{aligned}$$

$$\Rightarrow x_1 = \frac{c_1 - (a_{12}x_2 + \dots + a_{1n}x_n)}{a_{11}}$$

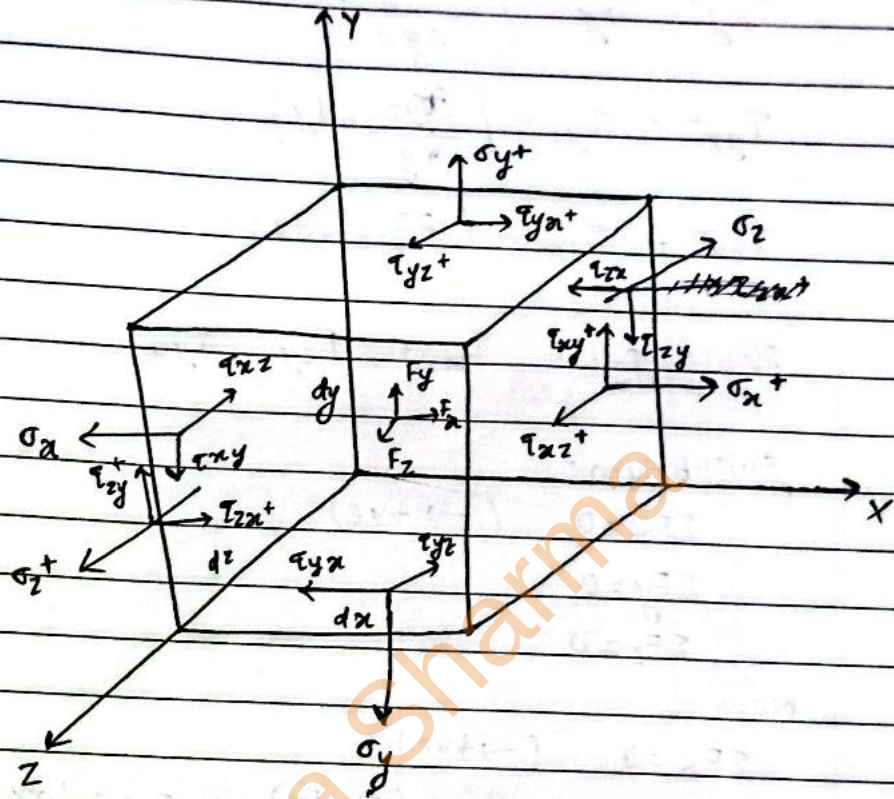
Now

$$x_1 = \frac{1}{a_{11}} \left[c_1 - \sum_{i=2}^n a_{1i}x_i \right]$$

$$x_2 = \frac{1}{a_{22}} \left[c_2 - \sum_{i=1, i \neq 2}^n a_{2i}x_i \right]$$

$$\& x_n = \frac{1}{a_{nn}} \left[c_n - \sum_{i=1, i \neq n}^n a_{ni}x_i \right]$$

EQUILIBRIUM EQUATION IN 3D



$\left\{ \begin{array}{l} \tau_{xy} \rightarrow (\text{कुन दिश}) \\ \uparrow \\ \text{कुन दिश मा lie गर्दै} \\ (\text{कुन plane मा}) \\ (\text{face}) \end{array} \right\}$

→ F_x, F_y & F_z are external forces per unit volume

→ Here,

$$\sigma_x^+ = \sigma_x + \left(\frac{\partial \sigma_x}{\partial x} \cdot dx \right)$$

$$\sigma_y^+ = \sigma_y + \left(\frac{\partial \sigma_y}{\partial y} \cdot dy \right)$$

$$\sigma_z^+ = \sigma_z + \left(\frac{\partial \sigma_z}{\partial z} \cdot dz \right)$$

Shear

$$\tau_{xy}^+ = \tau_{xy} + \left(\frac{\partial \tau_{xy}}{\partial x} \cdot dx \right)$$

$$\tau_{yz}^+ = \tau_{yz} + \left(\frac{\partial \tau_{yz}}{\partial y} \cdot dy \right)$$

$$\tau_{zx}^+ = \tau_{zx} + \left(\frac{\partial \tau_{zx}}{\partial z} \cdot dz \right)$$

Similarly... $\tau_{xz}^+, \tau_{zy}^+, \tau_{yx}^+$

Equilibrium:

$$\Sigma F_x = 0 \quad (\rightarrow +ve)$$

$$\Sigma F_y = 0$$

$$\Sigma F_z = 0$$

Now,

$$\Sigma F_x = 0 \quad (\rightarrow +ve)$$

$$\begin{aligned} \text{on } -\sigma_x (dy \cdot dz) + \sigma_x^+ (dy \cdot dz) + \tau_{yx} (dx \cdot dz) \\ - \tau_{yx}^+ (dx \cdot dz) + \tau_{zx} (dx \cdot dy) - \tau_{zx}^+ (dx \cdot dy) \\ + F_x \cdot dx \cdot dy \cdot dz = 0 \end{aligned}$$

$$\begin{aligned} \text{or, } (-\sigma_x dy dz) + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} \cdot dx \right) (dy \cdot dz) + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \cdot dy \right) (dx \cdot dz) \\ - \tau_{yx} (dx \cdot dz) + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \cdot dz \right) dx \cdot dy \\ - \tau_{zx} dx \cdot dy + F_x \cdot dx \cdot dy \cdot dz = 0 \end{aligned}$$

$$\text{or, } (\sigma_x + \sigma_x) dy dz + \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x \right) dx dy dz = 0$$

$$\text{or, } \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0 \quad \text{--- (1)}$$

$\tau_{xy} \rightarrow x$ dir & y change

Similarly,

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + F_z = 0$$

So, in matrix form,

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} + \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Symbolically,

$$[\sigma]^T \{\sigma\} + \{F\} = \{0\}$$

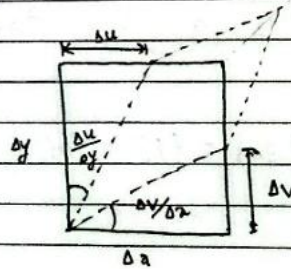
Strain-Displacement Relationship:



$$\epsilon_x = \frac{\Delta u}{\Delta x}$$

$$\epsilon_y = \frac{\Delta v}{\Delta y}$$

$$\text{Also, } \epsilon_z = \frac{\Delta w}{\Delta z}$$



$$\gamma_{xy} = \frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y}$$

(shear strain)

As, $\Delta x \rightarrow 0$ & $\Delta y \rightarrow 0$, strain-displacement relation, becomes,

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{yz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}; \quad \gamma_{zx} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

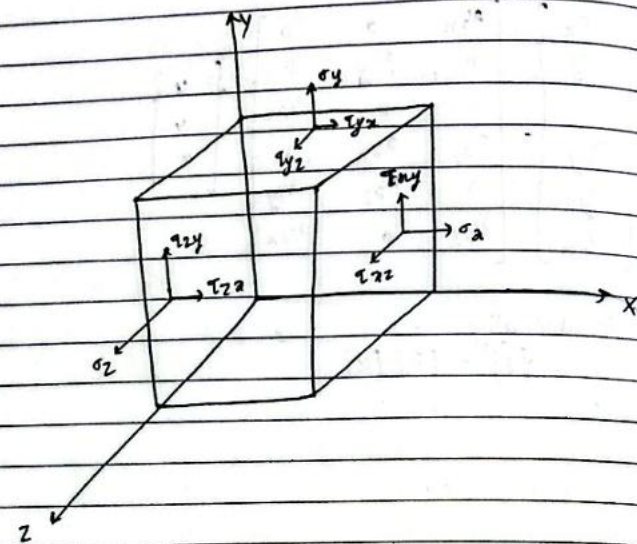
In matrix form:

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

Symbolically,

$$\{\epsilon\} = [\epsilon] \{u\}$$

Stress-Strain Constitutive Relationship



Normal strains,	Shear strains
$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E}$	$\gamma_{xy} = \frac{\tau_{xy}}{G}$
$\epsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_z}{E}$	$\gamma_{yz} = \frac{\tau_{yz}}{G}$
$\epsilon_z = \frac{\sigma_z}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}$	$\gamma_{zx} = \frac{\tau_{zx}}{G}$

Also, $\gamma_{xy} = \frac{\tau_{xy}}{G} = \tau_{xy} \frac{2(1+\nu)}{E}$

In Matrix Form,

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

Now,

$$\begin{aligned} \epsilon_x &= \frac{1}{E} (\sigma_x + \nu \sigma_y - \nu \sigma_z - \nu \sigma_y - \nu \sigma_z) \\ &= \frac{1}{E} \{ (1+\nu) \sigma_x - \nu (\sigma_x + \sigma_y + \sigma_z) \} \\ &= \frac{1}{E} \{ (1+\nu) \sigma_x - \nu \sigma_0 \} \end{aligned}$$

where, $\sigma_0 = \sigma_x + \sigma_y + \sigma_z$

Similarly,

$$\epsilon_y = \frac{1}{E} \{ (1+\nu) \sigma_y - \nu \sigma_0 \}$$

$$\epsilon_z = \frac{1}{E} \{ (1+\nu) \sigma_z - \nu \sigma_0 \}$$

Volumetric strain: $\theta_E = \epsilon_x + \epsilon_y + \epsilon_z$

$$\theta_E = \frac{1}{E} \{ (1+\nu) (\sigma_x + \sigma_y + \sigma_z) - 3\nu \sigma_0 \}$$

$$= \frac{1}{E} \{ (1+\nu) \sigma_0 - 3\nu \sigma_0 \}$$

$$= \frac{1}{E} \{ \sigma_0 - 2\nu \sigma_0 \}$$

$$\theta_E = \frac{\sigma_0}{E} (1-2\nu) \rightarrow \text{Volumetric form of Hooke's Law}$$

Now, $\epsilon_x = \frac{1}{E} \{ (1+\nu) \sigma_x - \nu \sigma_0 \}$

$$\text{or } \epsilon_x = \frac{1}{E} \{ (1+\nu) \sigma_x - \nu \cdot \frac{\sigma_0}{(1-2\nu)} \}$$

$$\text{on } E \epsilon_x = (1+\nu) \sigma_x - \frac{\nu E \theta_E}{1-2\nu}$$

$$\text{on } \sigma_x = \frac{E}{1+\nu} \epsilon_x + \frac{\nu E \theta_E}{(1-2\nu)(1+\nu)}$$

$$\sigma_x = 2\mu \epsilon_x + \lambda \sigma_z$$

$$\left. \begin{aligned} \text{here, } \mu &= \frac{E}{2(1+\nu)} \\ \lambda &= \frac{\nu E}{(1-2\nu)(1+\nu)} \end{aligned} \right\} \text{Lame's constant}$$

$$\sigma_x = 2\mu \epsilon_x + \lambda \sigma_z$$

$$= \frac{E}{(1+\nu)} \epsilon_x + \frac{\nu E}{(1+\nu)(1-2\nu)} (\epsilon_x + \epsilon_y + \epsilon_z)$$

$$= \frac{E}{(1+\nu)(1-2\nu)} [\nu \epsilon_x + \nu \epsilon_y + \nu \epsilon_z + \epsilon_x - 2\nu \epsilon_x]$$

$$= \frac{E}{(1+\nu)(1-2\nu)} [\epsilon_x - \nu \epsilon_x + \nu \epsilon_y + \nu \epsilon_z]$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\epsilon_x + \nu \epsilon_y + \nu \epsilon_z]$$

Similarly,

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\epsilon_y + \nu \epsilon_x + \nu \epsilon_z]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\epsilon_z + \nu \epsilon_x + \nu \epsilon_y]$$

Now,

$$\tau_{xy} = \tau_{xy} \cdot \frac{E}{2(1+\nu)}$$

$$= \tau_{xy} \cdot \frac{E}{2(1+\nu)} \cdot (1-2\nu)$$

$$= \frac{E}{2(1+\nu)(1-2\nu)} \cdot \tau_{xy} (1-2\nu)$$

$$\{\sigma\} = [D] \{\epsilon\}$$

6x6

$$\tau_{yz} = \frac{E}{2(1+\nu)(1-2\nu)} \cdot \tau_{yz} (1-2\nu)$$

$$\tau_{zx} = \frac{E}{2(1+\nu)(1-2\nu)} \cdot \tau_{zx} (1-2\nu)$$

In Matrix Form,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

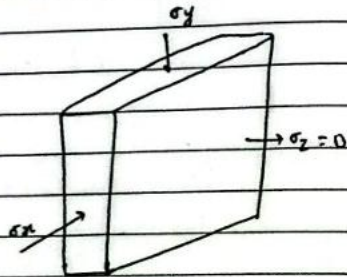
$$\text{Or, } \{\sigma\} = [D] \{\epsilon\}$$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix}$$

Plane Stress Condition

- A thin planar body subjected to in-plane loading is said to be in plane stress.
- Eg: Loading on wall along wall surface; Gussier plate
- In thin plate of uniform thickness, when stresses are acting in a plane of plate & they do not vary in direction perpendicular to plane, this condition is plane stress.

→ In this condition, $\sigma_z = 0$



$$\sigma_z = 0$$

$$\tau_{yz} = 0$$

$$\tau_{xz} = 0$$

$$\text{When } \tau_{yz} = 0 \Rightarrow \gamma_{yz} = 0$$

$$\tau_{xz} = 0 \Rightarrow \gamma_{xz} = 0$$

- Substituting the value of σ_z in stress-strain constitutive relationship.

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\epsilon_z + \nu\epsilon_x + \nu\epsilon_y]$$

$$\text{here, } \sigma_z = 0$$

$$\text{on } 0 = \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\epsilon_z + \nu\epsilon_x + \nu\epsilon_y]$$

$$\epsilon_z = \frac{-\nu}{(1-\nu)} (\epsilon_x + \epsilon_y)$$

Also, Using Lamé's constants

$$\sigma_z = \lambda \theta + 2\mu \epsilon_z$$

$$\begin{aligned} \rightarrow \sigma_x &= \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\epsilon_x + \nu\epsilon_y + \nu\epsilon_z] \\ &= \frac{E}{(1+\nu)(1-2\nu)} \left[(1-\nu)\epsilon_x + \nu\epsilon_y - \frac{\nu^2}{1-\nu} \epsilon_x - \frac{\nu^2}{1-\nu} \epsilon_y \right] \\ &= \frac{E}{(1-\nu^2)(1-2\nu)} [(1-\nu)^2 \epsilon_x + \nu(1-\nu)\epsilon_y - \nu^2 \epsilon_x - \nu^2 \epsilon_y] \\ &= \frac{E}{(1-\nu^2)(1-2\nu)} [\epsilon_x - 2\nu\epsilon_x + \nu^2 \epsilon_x + \nu\epsilon_y - \nu^2 \epsilon_y - \nu^2 \epsilon_y] \\ &= \frac{E}{(1-\nu^2)(1-2\nu)} [(1-2\nu)(\epsilon_x + \nu\epsilon_y)] \end{aligned}$$

$$\left[\sigma_x = \frac{E}{(1-\nu^2)} [\epsilon_x + \nu\epsilon_y] \right]$$

Similarly,

$$\sigma_y = \frac{E}{(1-\nu^2)} (\epsilon_y + \nu\epsilon_x)$$

$$\begin{aligned} \tau_{xy} &= G\gamma_{xy} = \frac{E}{2(1+\nu)} \gamma_{xy} \\ &= \frac{E}{2(1+\nu)} \cdot \frac{(1-\nu)}{(1-\nu)} \gamma_{xy} \end{aligned}$$

$$\left[\tau_{xy} = \frac{E(1-\nu)}{2(1-\nu^2)} \gamma_{xy} \right]$$

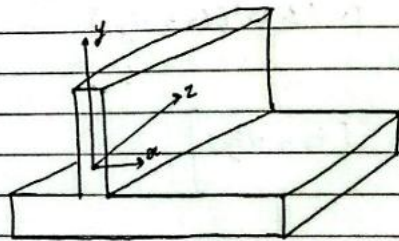
In Matrix form,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

constitutive matrix for plane stress condition.

Plane Strain Condition:-

- Out-of-plane loading
- strain in longitudinal direction is neglected
- $\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$



here, $\tau_{xz} = 0$

$\tau_{yz} = 0$

→ Stress-strain constitutive relationship becomes,

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\epsilon_x + \nu\epsilon_y]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\epsilon_y + \nu\epsilon_x]$$

$$\tau_{xy} = \frac{E}{(1+\nu)(1-2\nu)} [(0.5-\nu)\gamma_{xy}]$$

In matrix form,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & (0.5-\nu) \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Also,

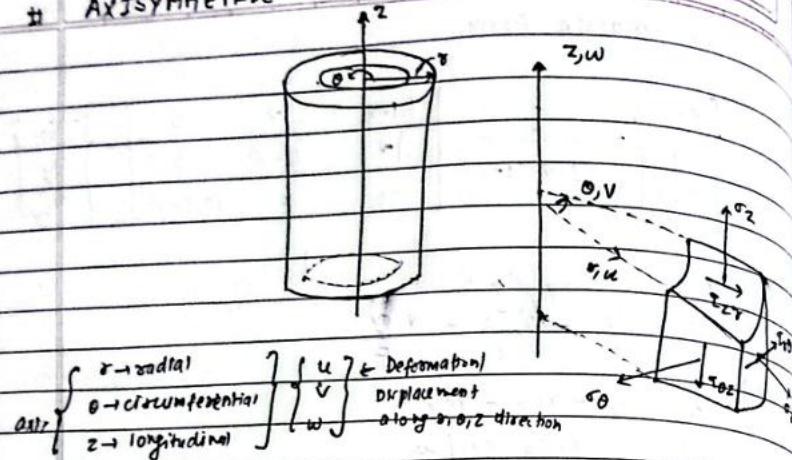
$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}\sigma_x - \frac{\nu}{E}\sigma_y$$

$$\therefore 0 = \sigma_z - \nu\sigma_x - \nu\sigma_y$$

$$\text{or } [\sigma_z = \nu(\sigma_x + \sigma_y)] \text{ u}$$

→ Example: If loading in the long body such as retaining wall, gravity wall or long cylinder do not vary significantly in the longitudinal direction

II AXISYMMETRIC PROBLEM



- A problem in which the geometry, loading, boundary conditions and the materials are symmetric with respect to an axis.
- It can be generated by revolving a plane area about an axis in the same plane.
- Example: Thick open-ended cylinder with internal pressure

→ Radial strain

$$\epsilon_r = \frac{\partial u}{\partial r}$$

→ Tangential strain (circumferential)

$$\begin{aligned} \epsilon_\theta &= \frac{\text{change in circumference}}{\text{original circumference}} \\ &= \frac{2\pi(r+u) - 2\pi r}{2\pi r} \\ &= \frac{u}{r} \end{aligned}$$

→ Longitudinal strain, $\epsilon_z = \frac{\partial w}{\partial z}$

→ Shear strain, $\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$

In matrix form,

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & (0.5-\nu) \end{bmatrix} \begin{Bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_\theta \\ \gamma_{rz} \end{Bmatrix}$$

Constitutive relation for axisymmetric

Strain displacement relationship:

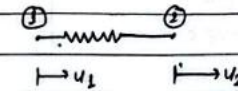
$$\begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \epsilon_z \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} \partial/\partial r & 0 \\ 1/r & 0 \\ 0 & \partial/\partial z \\ \partial/\partial z & \partial/\partial r \end{bmatrix} \begin{Bmatrix} u \\ w \end{Bmatrix}$$

Chapter-4

Finite Element Method (FEM)

Direct Stiffness Method

BAR ELEMENT



①, ② → nodes

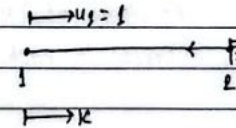
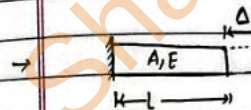
Stiffness matrix $[K]$

$$[K] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}_{2 \times 2}$$

$k \rightarrow$ Force per unit displacement

$k_{11} \rightarrow$ Force @ ① when unit deflection @ ①

$$= \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$



$$\Delta = \frac{PL}{AE}$$

$$P = \frac{\Delta AE}{L}$$

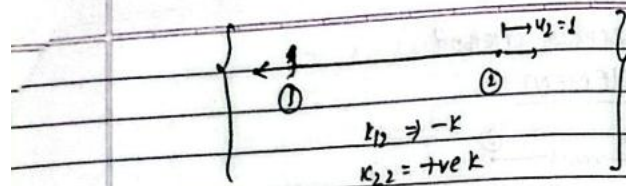
$$k = \frac{AE}{L} \quad \text{[If } \Delta = 1, P = k \text{]}$$

$$\begin{cases} k_{11} \rightarrow +ve \cdot k \\ k_{21} \rightarrow -ve \cdot k \end{cases}$$

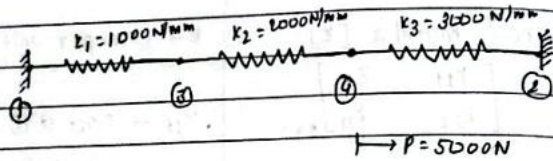
$$\therefore [K] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K] \{u\} = \{F\}$$

$$\text{Or } \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$



{Q}



Find the displacement at nodes ③ & ④
Reactions at nodes ① & ②
Force in each spring.

→ Solution:-
DOF = 4

Step ① Stiffness matrix

{ Local element } → Individual element stiffness matrix
{ Global }

Element ① ① ③ { For element ① ① & ③ are nodes }
 $[K_1] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix}$$

Element ② ③ ④
 $[K_2] = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix}$

Element ③ ④ ②
 $[K_3] = \begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix}$

Global Stiffness matrix

$$[K] = \begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 3000 & -2000 & 0 \\ 0 & -2000 & 5000 & -3000 \\ 0 & 0 & -3000 & 3000 \end{bmatrix}$$

Step ② Displacement Matrix

$$\{u\} = \begin{bmatrix} u_1 \\ u_3 \\ u_4 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ u_3 \\ u_4 \\ 0 \end{bmatrix}$$

Step ③ Force Matrix:

$$\{F\} = \begin{bmatrix} F_1 \\ F_3 \\ F_4 \\ F_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ 0 \\ 5000 \\ R_2 \end{bmatrix}$$

Step ④

$$[K]\{u\} = \{F\}$$

$$\begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 3000 & -2000 & 0 \\ 0 & -2000 & 5000 & -3000 \\ 0 & 0 & -3000 & 3000 \end{bmatrix} \begin{bmatrix} 0 \\ u_3 \\ u_4 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 \\ 0 \\ 5000 \\ R_2 \end{bmatrix}$$

Equations

$$1000 \times 0 - 1000 \times u_3 + 0 + 0 = R_1$$

$$-1000 \times 0 + 3000 \times u_3 - 2000 \times u_4 + 0 = 0$$

$$0 - 2000 \times u_3 + 5000 \times u_4 + 0 = 5000$$

$$0 + 0 - 3000 \times u_4 + 3000 \times 0 = R_2$$

u_3 & u_4 can be solved from here

u_3 & u_4 exists here only

(Can solve simultaneously to find u_3 & u_4)

R1 & R2 → unknown

Writing matrix in reduced form (to solve u_3 & u_4)
Reduced Matrix

$$\begin{bmatrix} 3000 & -2000 \\ -2000 & 5000 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 5000 \end{Bmatrix}$$

$$\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 5000 \end{Bmatrix} \begin{bmatrix} 3000 & -2000 \\ -2000 & 5000 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 5000 \end{Bmatrix}$$

On solving,

$$\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0.91 \\ 1.36 \end{Bmatrix} \text{ mm}$$

Thus,

For R_1 & R_2

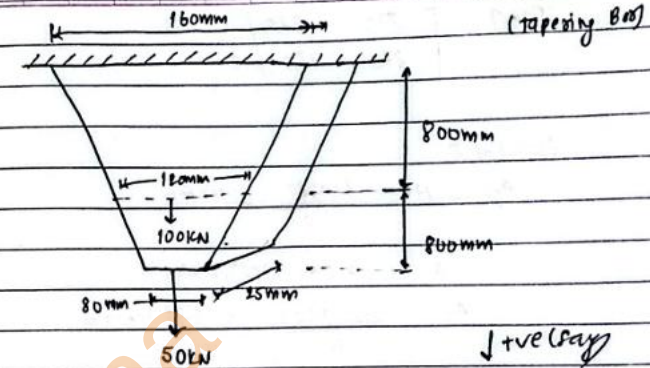
$$\textcircled{1} \quad -1000 \times u_3 = R_1$$

$$\text{or } R_1 = -1000 \times 0.91 = -910 \text{ N}$$

$$-3000 \times u_4 = R_2$$

$$\text{or } R_2 = -3000 \times 1.36 = -4080 \text{ N}$$

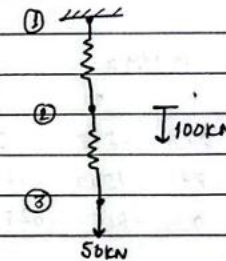
(Q.2)



$$E = 200 \text{ kN/mm}^2$$

Sp. wt. $\gamma = 78 \text{ kN/m}^3$ (do need to consider self wt. or not)

→ Spring Analogy:



Step 1: Stiffness Matrix:

Element ①

$$[k_1] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (\text{Axial force matrix } E \rightarrow \frac{AE}{L})$$

$$\star \text{ Avg. width} = \frac{160 + 120}{2} = 140 \text{ mm}$$

$$\text{Area} = A = 140 \times 25 = 3500 \text{ mm}^2$$

$$[k_1] = \frac{3500 \times 200}{1600} = 4375 \text{ kN/mm}$$

$$[K_1] = \begin{bmatrix} 875 & -875 \\ -875 & 875 \end{bmatrix}$$

Element ②

$$\text{Avg. width} = \frac{120+10}{2} = 100 \text{ mm}$$

$$A = 100 \times 25 = 2500 \text{ mm}^2$$

$$\frac{AE}{L} = \frac{2500 \times 200}{800} = 625 \text{ kN/mm}$$

$$[K_2] = \begin{bmatrix} 625 & -625 \\ -625 & 625 \end{bmatrix}$$

Global stiffness matrix:

$$[K] = \begin{bmatrix} 875 & -875 & 0 \\ -875 & 1500 & -625 \\ 0 & -625 & 625 \end{bmatrix}$$

Step 2: Displacement matrix:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix}$$

Step 3: Force Matrix:

↳ Externally applied
↳ Self wt.

$$F_{ext} = \begin{Bmatrix} 0 \\ +100 \\ +50 \end{Bmatrix}$$

↓ +ve

$$\{F\}_{\text{reaction}} = \begin{Bmatrix} R_1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{F\}_{\text{self wt.}} = \begin{Bmatrix} \\ \\ \end{Bmatrix}$$

$$\text{Self wt.} = \gamma \times V$$

$$= \gamma \times A \times L$$

$$= \gamma \times (w \times t) \times L$$

$$\begin{aligned} \text{Element ①} \Rightarrow \text{Self wt.} &= \gamma \times w \times t \times L \\ &= 78 \times 10^6 \text{ N/mm}^3 \times 140 \times 25 \times 800 \\ &= 0.218 \text{ kN} \end{aligned}$$

$$\begin{aligned} \text{Element ②} \Rightarrow \text{Self wt.} &= \gamma \times w \times t \times L \\ &= 78 \times 10^6 \times 100 \times 25 \times 800 \\ &= 0.156 \text{ kN} \end{aligned}$$

[FEM → self wt. nodes π divide $\frac{\text{Self wt.}}{\pi}$ eq. π th concept]

↳ Element ① self wt. each node π equally divide $\frac{\text{Self wt.}}{\pi}$

$$\{F_1\} = \begin{Bmatrix} 0.109 \\ 0.109 \end{Bmatrix} \quad \{F_2\} = \begin{Bmatrix} 0.078 \\ 0.078 \end{Bmatrix}$$

$$\{F\}_{\text{self wt.}} = \begin{Bmatrix} 0.109 \\ 0.109 + 0.078 \\ 0.078 \end{Bmatrix}$$

$$\{F\}_{\text{eff}} = \begin{Bmatrix} 0.109 \\ 0.187 \\ 0.078 \end{Bmatrix}$$

Step ①

$$[K]\{U\} = \{F\} \quad \{u_1 = 0 \text{ Boundary condition}\}$$

$$\rightarrow \begin{bmatrix} 875 & -875 & 0 \\ -875 & 1500 & -625 \\ 0 & -625 & 625 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} R_1 + 0.109 \\ 100 + 0.187 \\ 50 + 0.078 \end{Bmatrix}$$

Reduced matrix,

$$\begin{bmatrix} 1500 & -625 \\ -625 & 625 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 100.187 \\ 50.078 \end{Bmatrix}$$

$$\text{or, } \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1500 & -625 \\ -625 & 625 \end{bmatrix}^{-1} \begin{Bmatrix} 100.187 \\ 50.078 \end{Bmatrix}$$

On solving,

$$u_2 = 0.172 \text{ mm}$$

$$u_3 = 0.252 \text{ mm}$$

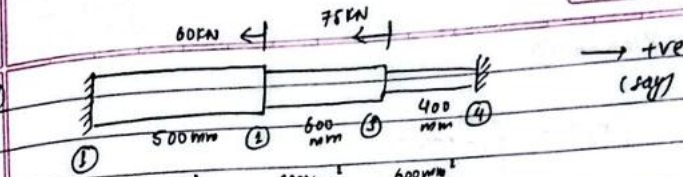
For L_1 ,

$$-875 \times u_2 = R_1 + 0.109$$

$$\Rightarrow u_2 = \frac{-R_1 - 0.109}{-875} = \frac{-R_1 - 0.109}{-875}$$

$$= -192.784 \text{ kN} \quad -150.61 \text{ kN}$$

Q3



$$\begin{array}{lll} \text{Area:} & 2400 \text{ mm}^2 & 1200 \text{ mm}^2 & 600 \text{ mm}^2 \\ E: & 83 \text{ GPa} & 70 \text{ GPa} & 200 \text{ GPa} \\ \alpha: & 18.9 \times 10^{-6} & 23 \times 10^{-6} & 11.7 \times 10^{-6} \text{ } ^\circ\text{C}^{-1} \end{array}$$

Increase in temp, $\Delta t = 80^\circ\text{C}$

Determine displacement, reactions & stresses.

sol:

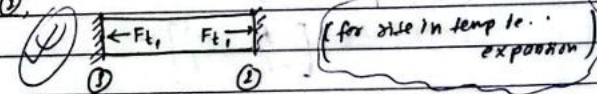
① Stiffness matrix

② Force matrix

$$\{F\} = \{F_{\text{ext}}\} + \{F_{\text{ext}}\} + \{F_{\text{temp}}\}$$

$$\left. \begin{array}{l} \Delta L_t = \alpha L \Delta t \\ \epsilon_t = \frac{\Delta L}{L} = \alpha \Delta t \\ \sigma = E \epsilon = E \alpha \Delta t \\ F_t = \sigma A = E \alpha \Delta t A \end{array} \right\}$$

Element ①



$$F_1 = \frac{2400 \times 83 \times 18.9 \times 10^{-6} \times 80}{10^3} = 0.3012 \text{ MN}$$

$$\{F_1\} = \begin{Bmatrix} -301.19 \\ 301.19 \end{Bmatrix} \text{ kN}$$

Element ①

$$\{F_2\} = \begin{Bmatrix} -154.56 \\ 154.56 \end{Bmatrix}$$

Element ②

$$\{F_3\} = \begin{Bmatrix} -112.32 \\ 112.32 \end{Bmatrix}$$

Thus

$$\{F\}_{\text{temp}} = \begin{Bmatrix} -301.19 \\ +301.19 - 154.56 \\ 154.56 - 112.32 \\ 112.32 \end{Bmatrix}$$

Stresses:

$$\sigma_1 = E_1 \cdot \frac{u_2 - u_1}{L} \quad \sigma_2 = E \cdot \frac{u_3 - u_2}{L} \quad \sigma_3 = E \cdot \frac{u_4 - u_3}{L}$$

1. Stiffness matrix:

Local: For Element ①

$$[k] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{2400 \times 83 \times 10^9 \times 10^{-3}}{500 (10^3)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 398.4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ KN/mm}$$

For Element ②

$$[k_2] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1200 \times 70 \times 10^9 \times (10^{-3})}{600 (10^3)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 140 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ KN/mm}$$

For Element ③

$$[k_3] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{600 \times 200 \times 10^9 \times (10^{-3})}{400 (10^3)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 300 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ KN/mm}$$

Global stiffness matrix:

$$[K] = \begin{bmatrix} 398.4 & -398.4 & 0 & 0 \\ -398.4 & 398.4 + 140 & -140 & 0 \\ 0 & -140 & 140 + 300 & -300 \\ 0 & 0 & -300 & 300 \end{bmatrix}$$

$$[K] = \begin{bmatrix} 398.4 & -398.4 & 0 & 0 \\ -398.4 & 538.4 & -140 & 0 \\ 0 & -140 & 440 & -300 \\ 0 & 0 & -300 & 300 \end{bmatrix} \text{ KN/mm}$$

2 Force Matrix:
 $\{F\} = \{F\}_{ext} + \{F\}_{exn} + \{F\}_{temp} \quad (\rightarrow +ve)$

$$\{F\}_{ext} = \begin{Bmatrix} 0 \\ -60 \\ -75 \\ 0 \end{Bmatrix}$$

$$\{F\}_{exn} = \begin{Bmatrix} R_1 \\ 0 \\ 0 \\ R_4 \end{Bmatrix}$$

F_{temp}
 Element ①

$$\left[\begin{array}{c} \leftarrow F_1 \\ F_1 \rightarrow \end{array} \right]$$

$$\begin{aligned} F_1 &= \sigma \times A = \epsilon \times E \times A \\ &= \frac{\Delta L}{L} \times E \times A \\ &= \alpha \Delta t \times E \times A \\ &= 18.9 \times 10^{-6} \times 80 \times 83 \times 10^9 \times 2400 \\ &\quad (10^3)^2 \\ &= 301.19 \text{ KN} \end{aligned}$$

Similarly

Element ②, $F_{t2} = \frac{23 \times 10^{-6} \times 80 \times 70 \times 10^9 \times 1200}{10^6}$
 $= 154.56 \text{ KN}$

Element ③, $F_{t3} = \frac{11.7 \times 10^{-6} \times 80 \times 700 \times 10^9 \times 600}{10^6}$
 $= 112.32 \text{ KN}$

$$\{F_{temp}\} = \begin{Bmatrix} -301.19 \\ 301.19 - 154.56 \\ 154.56 - 112.32 \\ 112.32 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$= \begin{Bmatrix} -301.19 \\ 146.63 \\ 42.24 \\ 112.32 \end{Bmatrix} \text{ KN}$$

$$\{F\} = \begin{Bmatrix} 0 + R_1 - 301.19 \\ -60 + 146.63 \\ -75 + 42.24 \\ 0 + R_4 + 112.32 \end{Bmatrix} = \begin{Bmatrix} R_1 - 301.19 \\ 86.63 \\ -32.76 \\ R_4 + 112.32 \end{Bmatrix} \text{ KN}$$

3 Displacement Matrix:

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} \quad \text{(Applying Boundary condition for fixed support)}$$

4

$$[K] \{u\} = \{F\}$$

$$\text{Or, } \begin{bmatrix} 398.4 & -398.4 & 0 & 0 \\ -398.4 & 538.4 & -140 & 0 \\ 0 & -140 & 440 & -300 \\ 0 & 0 & -300 & 300 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} R_1 - 301.19 \\ 86.63 \\ -32.76 \\ R_4 + 112.32 \end{Bmatrix}$$

Reduced Matrix

$$\begin{bmatrix} 538.4 & -140 \\ -140 & 440 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 86.63 \\ -32.76 \end{Bmatrix}$$

$$\text{or, } \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 538.4 & -140 \\ -140 & 440 \end{bmatrix}^{-1} \begin{Bmatrix} 86.63 \\ -32.76 \end{Bmatrix}$$

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.154 \\ -0.025 \end{Bmatrix} \text{ mm}$$

Now

$$-398.4 \times u_2 = R_1 - 301.19$$

$$\text{or, } R_1 = -398.4 \times 0.154 + 301.19$$

$$[R_1 = 239.84 \text{ kN}]$$

$$\text{Also, } -300 \times u_3 = R_4 + 112.32$$

$$\text{or, } R_4 = -300 \times (-0.025) - 112.32$$

$$[R_4 = -104.82 \text{ kN}]$$

5. Stresses

$$\sigma_1 = E_1 \times \epsilon_1 = E_1 \times \frac{u_2 - u_1}{L} \quad \text{(Tensile or compressive)}$$

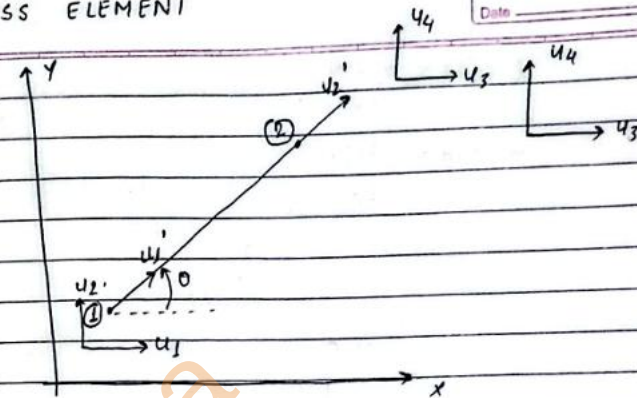
$$= 83 \times 10^9 \times \frac{0.154 - 0}{500}$$

$$= 0.0256 \text{ GPa}$$

$$\sigma_2 = E_2 \times \epsilon_2 = 70 \times 10^9 \times \frac{0.154 - (-0.025)}{600} = 0.0288 \text{ GPa}$$

$$\sigma_3 = E_3 \times \epsilon_3 = 200 \times 10^9 \times \frac{0 - 0.025}{400} = -0.0125 \text{ GPa}$$

II TRUSS ELEMENT



$u_1', u_2' \rightarrow$ local coordinate system displacement

$u_1, u_2, u_3, u_4 \rightarrow$ global coordinate system displacement

$$\{u'\} = \begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix}$$

From figure,

$$u_1' = u_1 \cos \theta + u_2 \sin \theta$$

$$u_2' = u_3 \cos \theta + u_4 \sin \theta$$

In matrix form,

$$\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

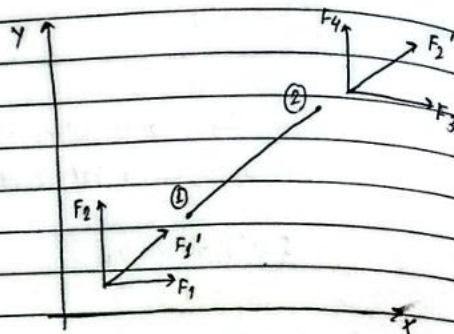
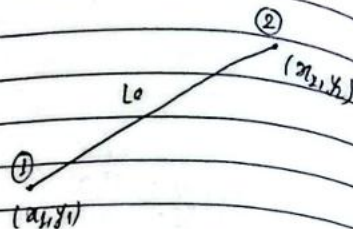
$$= \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\therefore \{u'\} = [T] \{u\}$$

Local $\uparrow$ $\uparrow$ Global
Displacement Transformation matrix

$$l = \cos \theta = \frac{x_2 - x_1}{L_e}$$

$$m = \sin \theta = \frac{y_2 - y_1}{L_e}$$



$$F_1 = F_1' \cos \theta \quad F_3 = F_3' \cos \theta$$

$$F_2 = F_1' \sin \theta \quad F_4 = F_2' \sin \theta$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \begin{Bmatrix} F_1' \\ F_2' \end{Bmatrix}$$

$$\begin{Bmatrix} F \end{Bmatrix} = [T]^T \begin{Bmatrix} F' \end{Bmatrix}$$

Global Local

STIFFNESS MATRIX

Local coordinate system:

$$[K'] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K'] \{u'\} = \{F'\} \quad \text{--- (1)}$$

$$\text{or, } [K'] [T] \{u\} = \{F'\}$$

Premultiplying by $[T]^T$ on both sides,

$$[T]^T [K'] [T] \{u\} = [T]^T \{F'\}$$

$$[K] \{u\} = \{F\}$$

Global coordinate system

Hence,

$$[K] = [T]^T [K'] [T]$$

$$[K] = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$= \frac{AE}{L} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

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Q.1

$E = 10^9 \text{ N/cm}^2$
 $A = 1.5 \text{ cm}^2$

$\Delta \rightarrow$ Element no

Solⁿ:

1 **Stiffness matrix**

Element ①:

$$[K_1] = \frac{AE}{L} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$l = 4\text{m}, m = 3\text{m}$
 $l^2 = 16\text{m}^2, m^2 = 9\text{m}^2$
 $lm = 12\text{m}^2$

$$[K_1] = \frac{1.5 \times 10^9}{4} \begin{bmatrix} 16 & 12 & -16 & -12 \\ 12 & 9 & -12 & -9 \\ -16 & -12 & 16 & 12 \\ -12 & -9 & 12 & 9 \end{bmatrix}$$

$$[K_1] = 10^3 \begin{bmatrix} 375 & 270 & -375 & -270 \\ 270 & 225 & -270 & -225 \\ -375 & -270 & 375 & 270 \\ -270 & -225 & 270 & 225 \end{bmatrix}$$

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Element ②:

$l = 4\text{m}, m = 3\text{m}$
 $l^2 = 16\text{m}^2, m^2 = 9\text{m}^2$
 $lm = 12\text{m}^2$

$$[K_2] = 10^3 \begin{bmatrix} 375 & 0 & -375 & 0 \\ 0 & 0 & 0 & 0 \\ -375 & 0 & 375 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Global stiffness matrix:

	1	2	3	4	5	6	
$[K]$	375	270	0	0	-375	-270	1
	270	225	0	0	-270	-225	2
	0	0	375	0	-375	0	3
	0	0	0	0	0	0	4
	-375	-270	-375	0	750	540	5
	-270	-225	0	0	540	450	6

2 **Displacement Matrix:**

$$\{u\} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_5 \\ u_6 \end{bmatrix}$$

(Boundary condition)

3 **Force matrix:**

$$\{F\} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ 500 \\ 300 \end{bmatrix}$$

4 $[k]\{u\} = \{F\}$

$$\begin{bmatrix} 132.6 & 132.6 & 0 & 0 & -132.6 & -132.6 \\ 132.6 & 132.6 & 0 & 0 & -132.6 & -132.6 \\ 0 & 0 & 375 & 0 & -375 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -132.6 & -132.6 & -375 & 0 & 507.6 & 132.6 \\ -132.6 & -132.6 & 0 & 0 & 132.6 & 132.6 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 500 \\ 300 \end{Bmatrix}$$

Reduction Matrix.

$$10^3 \begin{bmatrix} 507.6 & 132.6 \\ 132.6 & 132.6 \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 500 \\ 300 \end{Bmatrix}$$

$$\text{or } \begin{Bmatrix} u_5 \\ u_6 \end{Bmatrix} = \frac{1}{1000} \begin{bmatrix} 507.6 & 132.6 \\ 132.6 & 132.6 \end{bmatrix}^{-1} \begin{Bmatrix} 500 \\ 300 \end{Bmatrix}$$

$$\begin{Bmatrix} u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0.000533 \text{ cm} \\ 0.00172 \text{ cm} \end{Bmatrix}$$

Further

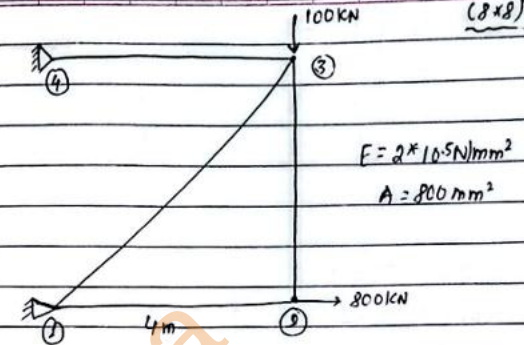
$$-375 \times u_5 = F_3 = -375 \times 0.000533 = -0.2 \text{ N} \rightarrow 200 \text{ N}$$

$$F_4 = 0$$

$$F_1 = -132.6 \times 0.000533 + (-132.6) \times (0.00172) = -0.3 \text{ N} = -300 \text{ N}$$

$$F_2 = -132.6 \times 0.000533 + (-132.6) \times (0.00172) = -0.3 \text{ N} \rightarrow 300 \text{ N}$$

(Q2)



(Q3) LOGAN Example 3.7 page 190

To illustrate how we can combine spring and bar elements in one structure, we now solve the two-bar truss supported by a spring shown in Figure below. Both bars have $E = 210 \text{ GPa}$ and $A = 5.0 \times 10^{-4} \text{ m}^2$. Bar one has a length of 5m and bar two has a length of 10m. The spring stiffness is $k = 20000 \text{ N/m}$.

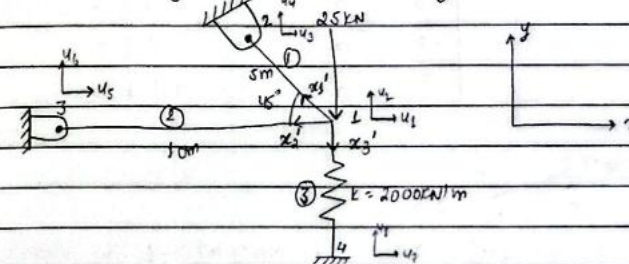


Fig. two-bar truss with spring support

1 Stiffness matrix $\theta = 9 = 135^\circ$

$$\text{Element 1 } l = \cos(135^\circ) = -0.707, \quad m = \sin(135^\circ) = 0.707$$

$$[k_1] = \frac{AE}{L} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

$$= 105 \times 10^3 \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix} \text{ kN/m}$$

Element 2

$\theta = \psi = 180^\circ$
 $k = \cos(180^\circ) = -1$, $m = \sin(180^\circ) = 0$

Replacing $\frac{AE}{L}$ with spring constant 'K'
 $[K_2] = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$[K_2] = 105 \times 10^3 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix}$$

Element 3

$\theta = \psi = 270^\circ$
 $k = \cos(270^\circ) = 0$, $m = \sin(270^\circ) = -1$

Replacing $\frac{AE}{L}$ with spring constant 'K'

$$[K_3] = 2000 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{matrix} u_5 \\ u_6 \\ u_7 \\ u_8 \end{matrix} \text{ kN/m}$$

Global stiffness matrix

$$[K] = \begin{bmatrix} 210 & -105 & -105 & 105 & 0 & 0 & 0 & 0 \\ -105 & 105 & 105 & -105 & 0 & 0 & 0 & 0 \\ -105 & 105 & 105 & -105 & 0 & 0 & 0 & 0 \\ 105 & -105 & -105 & 105 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 105 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 200 & 0 & -20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20 & 0 & 20 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{matrix} \times 10^3 \text{ kN/m}$$

Force matrix

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -25 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Displacement matrix

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix}$$

Applying Boundary condition

4

$$[K]\{u\} = \{F\}$$

Reduced Matrix:

$$10^3 \begin{bmatrix} 210 & -105 \\ -105 & 105 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -25 \end{Bmatrix}$$

Solving, $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -1.724 \times 10^{-3} \\ -3.448 \times 10^{-3} \end{Bmatrix} \text{ m}$

5

Stress in bar element

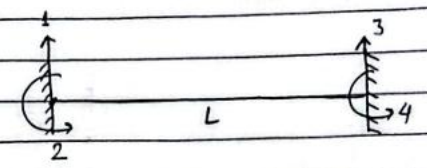
$$\sigma_1 = \frac{E}{L} [0.707 \quad -0.707 \quad -0.707 \quad 0.707] \begin{Bmatrix} -1.724 \times 10^{-3} \\ -3.448 \times 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

$$= 51.2 \text{ MPa (T)}$$

'OF' $\sigma_2 = \frac{210 \times 10^6}{10} * (-1.724 \times 10^{-3} - 0) = -36.2 \text{ MPa}$
[36.2 MPa (C)]

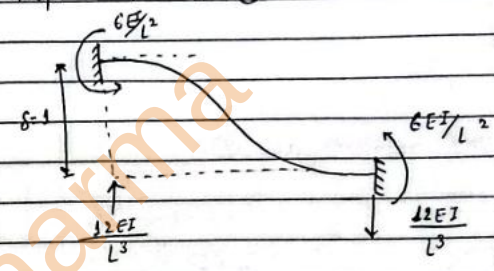
Surendra Sharma

Beam Element



Actual 6 DOF
But on beam
only vertical
forces exist &
beam is considered
inextensible so
DOF along
axial direction
is not taken

Unit displacement at (1):



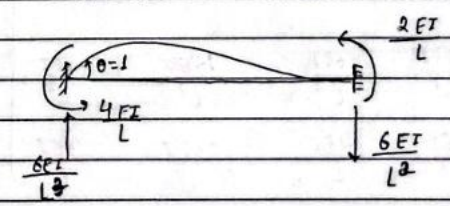
$$K_{11} = \frac{12EI}{L^3}$$

$$K_{21} = \frac{6EI}{L^2}$$

$$K_{31} = \frac{12EI}{L^3}$$

$$K_{41} = \frac{6EI}{L^2}$$

Unit rotation at (2):



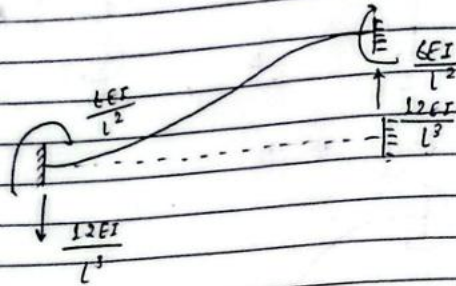
$$K_{12} = \frac{6EI}{L^2}$$

$$K_{22} = -\frac{6EI}{L^2}$$

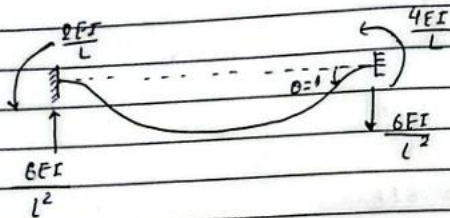
$$K_{32} = \frac{4EI}{L}$$

$$K_{42} = \frac{2EI}{L}$$

Unit displacement at (3).



Unit rotation at (4).



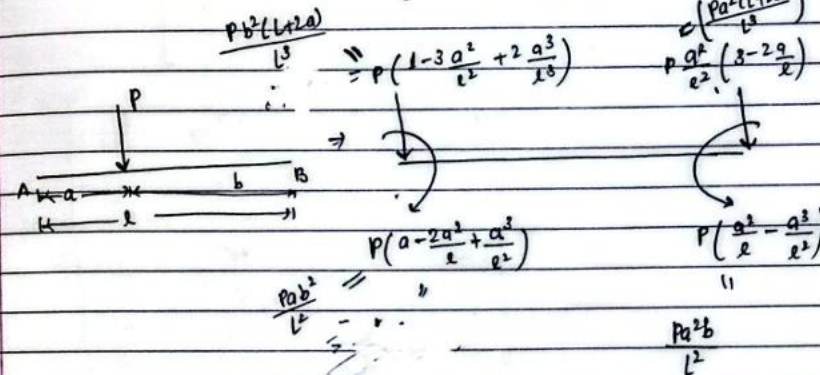
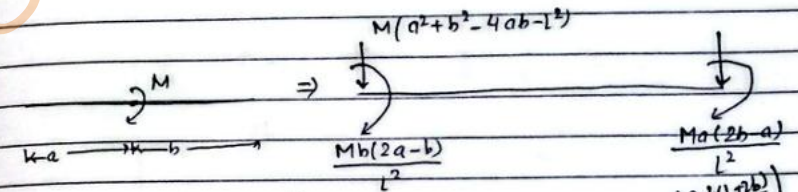
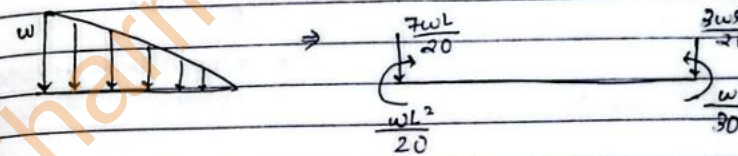
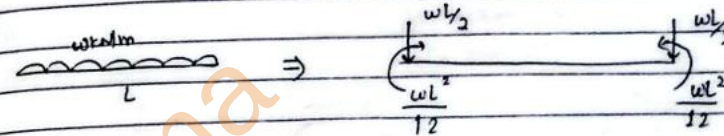
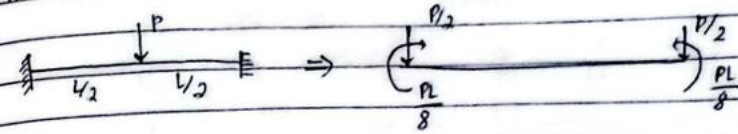
∴ Stiffness Matrix is:

$$[k] = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

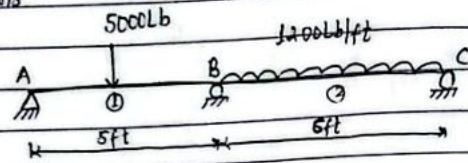
$$G = \frac{E}{2(1+\nu)}$$

NODAL LOADS

Loads in beam are transferred to nodes as equivalent nodal loads



(9) Determine the nodal displacement and reaction at the supports



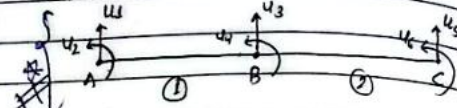
$E = 30 \times 10^6 \text{ psi}$

$I = 305 \text{ in}^4$

Solⁿ:

6 DOF

① Stiffness matrix



(↑ → vertical displacement) → u_1, u_3, u_5

(↺ → rotation) → u_2, u_4, u_6

Element ①

$$[k]_1 = EI \begin{bmatrix} \frac{12}{5^3} & \frac{6}{5^2} & -\frac{12}{5^3} & \frac{6}{5^2} \\ \frac{6}{5^2} & \frac{4}{5} & -\frac{6}{5^2} & \frac{2}{5} \\ -\frac{12}{5^3} & -\frac{6}{5^2} & \frac{12}{5^3} & -\frac{6}{5^2} \\ \frac{6}{5^2} & \frac{2}{5} & -\frac{6}{5^2} & \frac{4}{5} \end{bmatrix}$$

$$= 9.15 \times 10^9 \begin{bmatrix} \frac{12}{(5 \times 12)^3} & \frac{6}{(5 \times 12)^2} & -\frac{12}{(5 \times 12)^3} & \frac{6}{(5 \times 12)^2} \\ \frac{6}{(5 \times 12)^2} & \frac{4}{(5 \times 12)} & -\frac{6}{(5 \times 12)^2} & \frac{2}{(5 \times 12)} \\ -\frac{12}{(5 \times 12)^3} & -\frac{6}{(5 \times 12)^2} & \frac{12}{(5 \times 12)^3} & -\frac{6}{(5 \times 12)^2} \\ \frac{6}{(5 \times 12)^2} & \frac{2}{(5 \times 12)} & -\frac{6}{(5 \times 12)^2} & \frac{4}{(5 \times 12)} \end{bmatrix}$$

$= 10^4$	50.83	152.5	-50.83	152.5	1
	152.5	61000	-152.5	30500	2
	-50.83	-152.5	50.83	-152.5	3
	152.5	30500	-152.5	61000	4

Element ②

$[k]_2 = 10^4$	29.412	1058.83	-29.412	1058.83	3
	1058.83	50223.94	-1058.83	25111.97	4
	-29.412	-1058.83	29.412	-1058.83	5
	1058.83	25111.97	-1058.83	50223.94	6

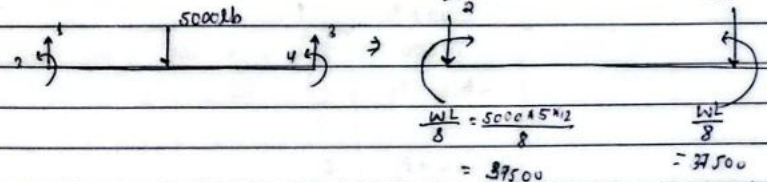
Global stiffness matrix

$[K] = 10^4$	1	2	3	4	5	6	
	50.83	152.5	-50.83	152.5	0	0	1
	152.5	61000	-152.5	30500	0	0	2
	-50.83	-152.5	50.83 + 29.412	-152.5 + 1058.83	1058.83	0	3
	152.5	30500	-152.5 + 1058.83	30500 + 25111.97	-1058.83	25111.97	4
	0	0	-29.412	-1058.83	29.412	-1058.83	5
	0	0	1058.83	25111.97	-1058.83	50223.94	6

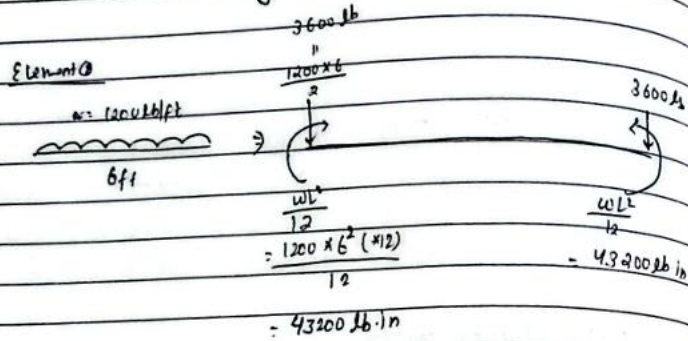
⑦ Force Matrix:

Nodal loads:

Element ①



$$\{F_1\} = \begin{Bmatrix} -2500 \\ -37500 \\ -2500 \\ 37500 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \quad \begin{matrix} (+ve) \\ (-ve) \end{matrix}$$



$$\{F_2\} = \begin{Bmatrix} -3600 \\ -43200 \\ -3600 \\ +43200 \end{Bmatrix} \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

Global force matrix:

$$\{F\} = \begin{Bmatrix} -2500 + R_1 \\ -37500 + 0 \\ -2500 - 3600 + R_2 \\ 37500 - 43200 + 0 \\ -3600 + R_3 \\ 43200 + 0 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

$$= \begin{Bmatrix} -2500 + R_1 \\ -37500 \\ -6100 + R_2 \\ -5700 \\ -3600 + R_3 \\ 43200 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

③ Displacement Matrix:

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} \quad \text{Applying Boundary condition} \quad = \begin{Bmatrix} 0 \\ u_2 \\ 0 \\ u_4 \\ 0 \\ u_6 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

④ Equation:

$$[K]\{u\} = \{F\}$$

Apply Boundary condition
Reduced Matrix:

$$10^4 \begin{bmatrix} 61000 & 30500 & 0 \\ 30500 & 11223.94 & 25411.97 \\ 0 & 25411.97 & 50823.94 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} -37500 \\ -5700 \\ 43200 \end{Bmatrix}$$

$$\text{or, } \begin{Bmatrix} u_2 \\ u_4 \\ u_6 \end{Bmatrix} = \frac{1}{10^4} \begin{bmatrix} 61000 & 30500 & 0 \\ 30500 & 11223.94 & 25411.97 \\ 0 & 25411.97 & 50823.94 \end{bmatrix} \begin{Bmatrix} -37500 \\ -5700 \\ 43200 \end{Bmatrix}$$

$$u_2 = -5.638 \times 10^{-5} \text{ rad.}$$

$$u_4 = -1.019 \times 10^{-5} \text{ rad.}$$

$$u_6 = +9.010 \times 10^{-5} \text{ rad.}$$

R

$$0 \times (1525 \times u_2 + 1525 \times u_4 + 0 \times u_6) = -2500 + R_1$$

$$\text{or, } R_1 = 2489.9 \approx 2500 \text{ lb} \quad 1484.8 \text{ lb}$$

$$10^3 [-152 \times u_2 + 461 \times u_4 + 1082 \times u_6] = -6100 + F_2$$

$$\text{or } F_2 = 6399.24 \approx 3000 + 7561.34$$

$$10^3 [-1082 \times u_4 + 107 \times u_6] = -3600 + F_3$$

$$\text{or } F_3 = 2713.49 \text{ kN}$$

- 1) Shape function/Interpolation function.
- In FEM, field variables at nodal points are found by rigorous analysis.
 - But field variable at any point inside the element is function of values at nodal points.
 - This function which relates field variable at any point within the element with the field variable of nodal point is called shape function.
 - In two dimensional stress analysis, where basic variable is displacement is given by:

$$u = \sum N_i u_i$$

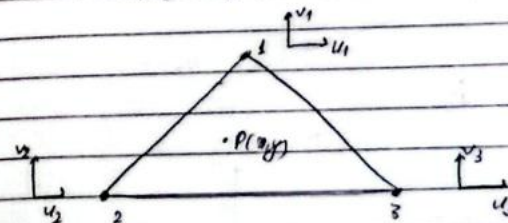
$$v = \sum N_i v_i$$

where summation is over the number of nodes

- For example, for 3-noded triangular element, displacement at $P(x, y)$ is given by

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$



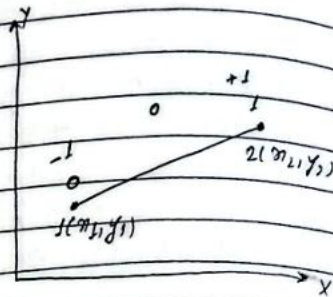
$$\text{or } \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$\Rightarrow \{u\} = [N] \{u_i\}$$

↗ Nodal displacement

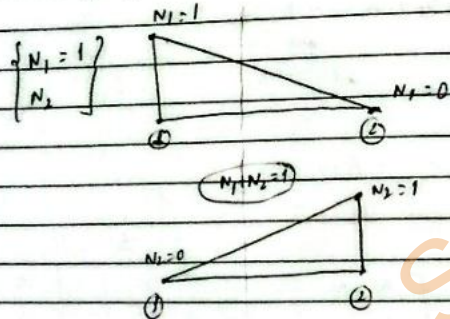
Natural coordinate system

→ It is unitless system



1 & 2 → Cartesian coordinate system
0 to 1 '0' -1 to 1 → Natural (unitless)
Let number of entry
(for some certain reqd. 4 entries, (x_1, y_1) & (x_2, y_2))

* Nodal weights

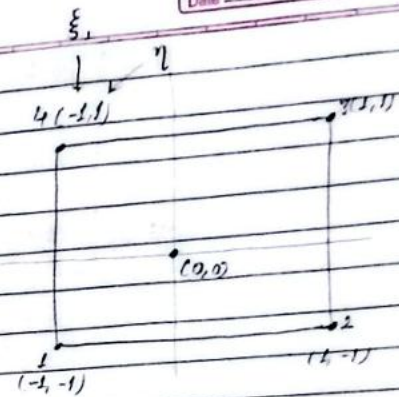


N_1 & N_2 → shape function → may be of any shape (here, linear)

→ At any pt. b/w, $N_1 + N_2 = 1$

* For plane,

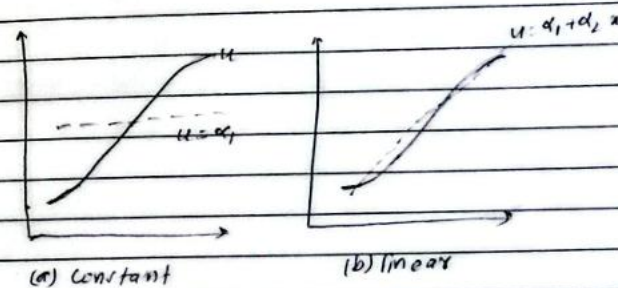
Denotation → (ξ, η)

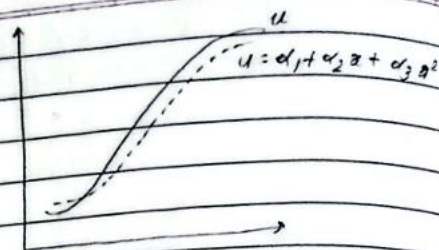


* Polynomial Function

→ one of the polynomial function are commonly used shape function because of following benefit

1. Easy to handle mathematically i.e. Differentiation and integration of polynomial function is easy.
2. Any variable can be approximated by using polynomial function. If the function is highly non-linear, we may have to approximate with higher order polynomial.





(c) Quadratic
(high degree of reliability)

One-dimensional polynomial shape function:

A general one-dimensional polynomial shape function of n^{th} order is given by

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \dots + \alpha_{n+1} x^n$$

$$\text{i.e. } u = [G] \{ \alpha \}$$

where,

$$[G] = [1 \quad x \quad x^2 \quad \dots \quad x^n]$$

$$\{ \alpha \}^T = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \dots \quad \alpha_{n+1}]$$

Two dimensional polynomial shape function

$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \dots + \alpha_m y^n$$

$$v(x, y) = \alpha_{m+1} + \alpha_{m+2} x + \alpha_{m+3} y + \alpha_{m+4} x^2 + \alpha_{m+5} xy + \dots + \alpha_{2m} y^n$$

$$\text{i.e. } \{ \delta \} = \{ u \} = [G] \{ \alpha \}$$

$$= \begin{bmatrix} G_1 & 0 \\ 0 & G_1 \end{bmatrix} \{ \alpha \}$$

where,

$$G_1 = [1 \quad x \quad y \quad x^2 \quad xy \quad y^2 \quad \dots \quad y^n]$$

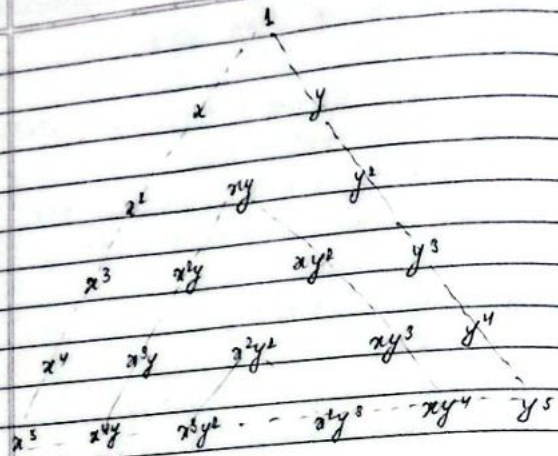
$$\{ \alpha \} = [\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_m \quad \alpha_{m+1} \quad \dots \quad \alpha_{2m}]^T$$

→ It may be observed that in 2D polynomial function, total number of terms (m) in each of n^{th} order polynomial is

or

$$m = \frac{(n+1)(n+2)}{2}$$

→ Another convenient way to remember complete 2-D polynomial terms is in form of PASCAL TRIANGLE

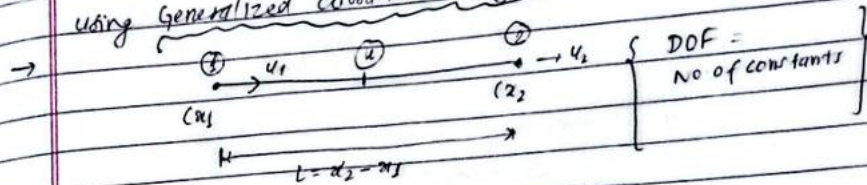


PASCAL TRIANGLE

Order	no. of terms
$u = \alpha_1$ constant	1
$u = \alpha_1 + \alpha_2 x + \alpha_3 y$	3
$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$	6
	15

touch

(Q) Find the shape function for a two noded bar element using Generalized coordinate system



here, nodal displacements are u_1 & u_2

Polynomial function is

$$u = \alpha_1 + \alpha_2 x \quad \text{--- (1)}$$

DOF = 2 ; No. of constants (α) = 2

Applying Boundary conditions:

At node ①

$$x = x_1 \quad \text{constant}$$

$$u = u_1$$

At node ②

$$x = x_2 \quad \text{constant}$$

$$u = u_2$$

from eqn (1)

$$u_1 = \alpha_1 + \alpha_2 x_1$$

$$u_2 = \alpha_1 + \alpha_2 x_2$$

In matrix form

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \text{--- (2)}$$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \frac{1}{x_2 - x_1} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \frac{1}{L} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{L} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Now,

$$u = d_1 + d_2 x$$

$$= \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$= \frac{1}{L} \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{L} \begin{bmatrix} (x_2 - x) & (-x_1 + x) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{x_2 - x}{L} & \frac{-x_1 + x}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u = N_1 u_1 + N_2 u_2$$

here,

$$\begin{cases} N_1 = \frac{x_2 - x}{L} \\ N_2 = \frac{-x_1 + x}{L} \end{cases} \Rightarrow \text{shape functions}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

determinant

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

② Shape function of bar element using natural coordinate system (ξ)



→ Soln:

Polynomial function is

$$u = \alpha_1 + \alpha_2 \xi$$

Applying B.C.

$$\xi = -1, u = u_1 \Rightarrow u_1 = \alpha_1 + \alpha_2(-1) = u_1 = \alpha_1 - \alpha_2$$

$$\xi = +1, u = u_2 \Rightarrow u_2 = \alpha_1 + \alpha_2(+1) = u_2 = \alpha_1 + \alpha_2$$

In matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

$$\text{or } \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

From (1),

$$u = \begin{bmatrix} 1 & \xi \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & \xi \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

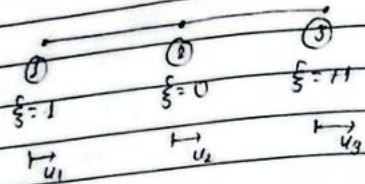
$$u = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

here,

$$\text{Shape functions: } N_1 = \frac{1-\xi}{2} ; N_2 = \frac{1+\xi}{2}$$

5) shape function of 3-noded bar element using natural coordinate system (ξ)



DOF = 3

u_1, u_2, u_3

→ Soln:

Polynomial function u

$$u = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2 \quad (1)$$

Applying B.C.

$$u_1 = \alpha_1 - \alpha_2 + \alpha_3$$

$$u_2 = \alpha_1 + 0 + 0$$

$$u_3 = \alpha_1 + \alpha_2 + \alpha_3$$

Matrix form

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = \begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \begin{bmatrix} -\frac{\xi+\xi^2}{2} & \frac{2-2\xi^2}{2} & \frac{\xi+\xi^2}{2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{\xi}{2}(\xi-1) & (1-\xi^2) & \frac{\xi}{2}(1+\xi) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

here

$$N_1 = \frac{\xi}{2}(\xi-1)$$

$$N_2 = 1 - \xi^2$$

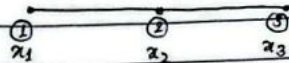
$$N_3 = \frac{\xi}{2}(1+\xi)$$

Shape function using Lagrange Polynomial

$$L(x) = \prod_{m=1, m \neq k}^n \frac{x - x_m}{x_k - x_m}$$

1-D

Example: 3-noded Bar element (1-D element)

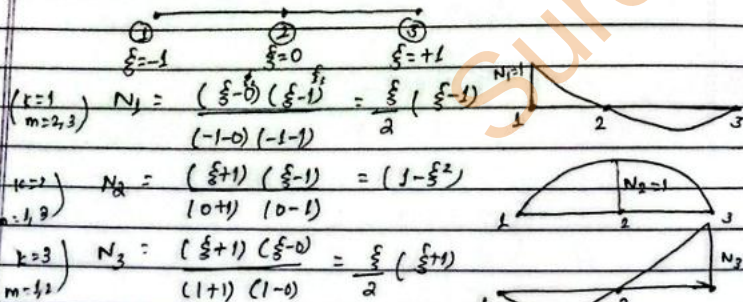


$$N_1 = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}$$

$$N_2 = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$N_3 = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

Example: 3-noded Bar element (in natural coordinate system)



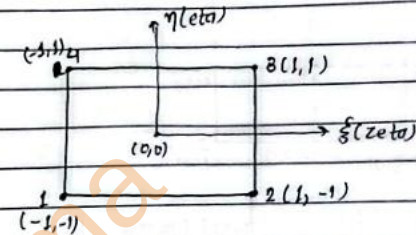
$$N_1 = \frac{(xi - 0)(xi - 1)}{(-1 - 0)(-1 - 1)} = \frac{xi(xi - 1)}{2}$$

$$N_2 = \frac{(xi + 1)(xi - 1)}{(0 + 1)(0 - 1)} = -(xi^2 - 1)$$

$$N_3 = \frac{(xi + 1)(xi - 0)}{(1 + 1)(1 - 0)} = \frac{xi(xi + 1)}{2}$$

2-D element (Rectangular)

(4-noded Rectangle Element)



$$N_i = \frac{1}{4} (1 \pm \xi_i) (1 \pm \eta_i)$$

$$\prod_{m=1}^4 \frac{x - x_m}{x_i - x_m}$$

$$N_1 = \frac{(xi - 1)(eta - 1)}{(1 - 1)(-1 - 1)} = \frac{1}{4} (xi - 1)(eta - 1)$$

or

$$\frac{(xi - xi_1)(eta - eta_1)}{(xi_2 - xi_1)(eta_2 - eta_1)}$$

$\xi \rightarrow$ Hz. (connect nodes 1 & 2)

$\eta \rightarrow$ Vt. (connect nodes 1 & 4)

$$N_2 = \frac{(xi - xi_1)(eta - eta_3)}{(xi_2 - xi_1)(eta_2 - eta_3)} = \frac{(xi + 1)(eta - 1)}{(1 + 1)(-1 - 1)}$$

$$= \frac{1}{4} (1 + xi)(eta - 1)$$

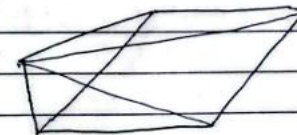
$$N_3 = \frac{1}{4} (1 + xi)(1 + eta)$$

$$N_4 = \frac{(xi - 1)(eta + 1)}{(1 - 1)(1 + 1)} = \frac{1}{4} (1 - xi)(1 + eta)$$

In general form,

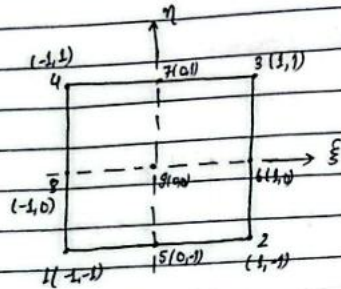
$$N_i = \frac{1}{4} (1 + \xi_i \xi) (1 + \eta_i \eta)$$

$N_1 = 1$



(Linear)

* 9-noded Rectangular Element:



$$N_1 = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} \times \frac{(\eta - \eta_2)(\eta - \eta_4)}{(\eta_1 - \eta_2)(\eta_1 - \eta_4)}$$

$$= \frac{(\xi - 0)(\xi - 1)}{(-1 - 0)(-1 - 1)} \times \frac{(\eta - 0)(\eta - 1)}{(-1 - 0)(-1 - 1)}$$

$$= \frac{1}{4} \times \xi \eta (\xi - 1)(\eta - 1)$$

Similarly,

$$N_2 = \frac{(\xi - 0)(\xi + 1)}{(1 - 0)(1 + 1)} \times \frac{(\eta - 0)(\eta - 1)}{(-1 - 0)(-1 - 1)}$$

$$= \frac{1}{4} \times \xi \eta (\xi + 1)(\eta - 1)$$

$$N_3 = \frac{\xi(\xi + 1)\eta(\eta + 1)}{4}$$

$$N_4 = \frac{\xi(\xi - 1)\eta(\eta + 1)}{4}$$

For corner nodes:

$$N_i = \frac{\xi \eta (\xi + \xi_i)(\eta + \eta_i)}{4}$$

$$N_5 = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} \cdot \frac{(\eta - \eta_1)(\eta - \eta_2)}{(\eta_3 - \eta_1)(\eta_3 - \eta_2)}$$

$$= \frac{(\xi^2 - 1)\eta(\eta - 1)}{-2}$$

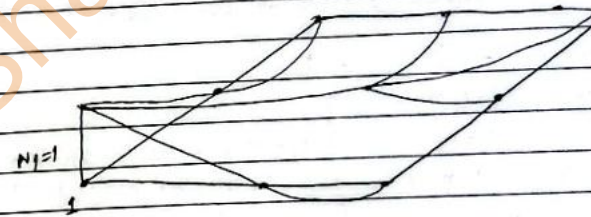
$$N_6 = \frac{\xi(\xi + 1)(\eta^2 - 1)}{-2}$$

$$N_7 = \frac{(\xi^2 - 0)(\xi(\xi + 1)(\xi - 1)) \cdot (\eta - 0)(\eta + 1)}{(0 + 1)(0 - 1)(1 - 0)(1 + 1)}$$

$$= \frac{(\xi^2 - 1)\eta(\eta + 1)}{-2}$$

$$N_8 = \frac{\xi(\xi - 1)(\eta^2 - 1)}{-2}$$

$$N_9 = \frac{(\xi^2 - 1)(\eta^2 - 1)}{4}$$

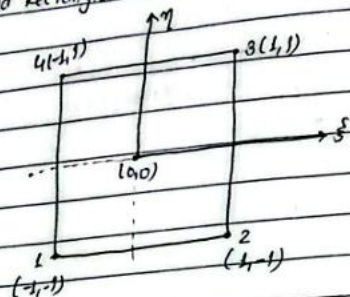


(parabolic nature)

Shape function for serendipity Element

Eg1

4-noded Rectangle element



$$\begin{cases} N_1 = 1 & \text{at node 1 } \xi = -1, \eta = -1 \\ N_1 = 0 & \text{along line 2-3, } \xi = 1 \text{ or } \xi - 1 = 0 \\ N_1 = 0 & \text{along line 3-4, } \eta = 1 \text{ or } \eta - 1 = 0 \end{cases}$$

{Now, make a equation satisfying all above condition}

$$N_1 = c(\xi + 1)(\eta + 1) \quad \text{--- (1)}$$

Node (1), $\xi = -1, \eta = -1$

$$N_1 = 1$$

Node (3), $\xi = 1, \eta = 1$

$$N_1 = 0$$

Node (2), $\xi = 1, \eta = -1$
 $\xi = -1, \eta = -1$

$$N_1 = 0$$

Eq (1) satisfies the condition for all other nodes i.e. ($N_1 = 0$ for other nodes)

Apply Boundary condition.

$$1 = c(-1-1)(-1-1) \quad (\text{for } N_1)$$

$$c = 1/4$$

$$\therefore N_1 = 1/4 (\xi + 1)(\eta + 1) \quad \underline{\underline{u}}$$

Similarly,

For N_2

$$\begin{cases} N_2 = 1 & \text{at Node (2) } \xi = 1, \eta = -1 \\ N_2 = 0 & \text{along line 1-4, } \xi = -1 \text{ or } \xi + 1 = 0 \\ N_2 = 0 & \text{along line 4-3, } \eta = 1 \text{ or } \eta - 1 = 0 \end{cases}$$

$$\therefore N_2 = c(\xi - 1)(\eta - 1)$$

Apply Boundary condition

$$1 = c(1-1)(-1-1)$$

$$c = \frac{1}{-4}$$

$$\therefore N_2 = \frac{1}{4} (\xi - 1)(\eta - 1) \quad \underline{\underline{u}}$$

For N_3

$$\begin{cases} N_3 = 1 & \text{at Node (3) } \xi = 1, \eta = 1 \\ N_3 = 0 & \text{along line 1-4, } \xi = -1 \text{ or } \xi + 1 = 0 \\ N_3 = 0 & \text{along line 1-2, } \eta = -1 \text{ or } \eta + 1 = 0 \end{cases}$$

$$\therefore N_3 = c(1 + \xi)(1 + \eta)$$

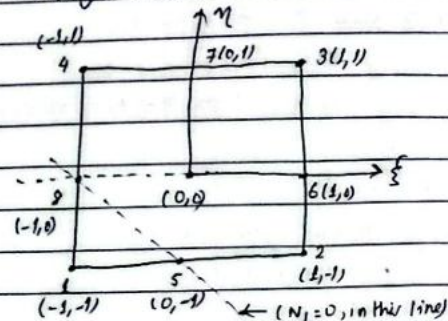
Apply Boundary condition

$$1 = c(2)(2) \Rightarrow c = 1/4$$

$$N_3 = 1/4 (1 + \xi)(1 + \eta) \quad \underline{\underline{u}}$$

For N_4

Eg (11) 8-noded rectangle element



$N_1 \Rightarrow$

$$\begin{aligned} N_1 &= 1 \text{ @ node (1) } \xi = -1/\eta = -1 \\ N_1 &= 0 \text{ along 3-4 line, } \eta = 1 \Rightarrow \eta - 1 = 0 \\ N_1 &= 0 \text{ along 2-3 line, } \xi = 1 \Rightarrow \xi - 1 = 0 \\ N_1 &= 0 \text{ along line 5-8, } \\ \eta - 0 &= \frac{-1 - 0}{0 + 1} (\xi + 1) \\ \eta &= -(\xi + 1) \\ (\xi + \eta + 1) &= 0 \end{aligned}$$

include
all
remaining
f-conditions
(nodes)

$$N_1 = c (\xi - 1) (\eta - 1) (\xi + \eta + 1)$$

Apply boundary condition

$$1 = c (-1 - 1) (-1 - 1) (-1 - 1 + 1)$$

$$c = -1/4$$

$$\therefore N_1 = -1/4 (\xi - 1) (\eta - 1) (\xi + \eta + 1)$$

Similarly,

$$N_5 = 1 \text{ when } \xi = 0, \eta = -1$$

$$N_5 = 0 \text{ when } \xi = 1 \text{ or } \xi - 1 = 0$$

$$N_5 = 0 \text{ when } \eta = 1 \text{ or } \eta - 1 = 0$$

$$N_5 = 0 \text{ when } \xi = -1 \text{ or } \xi + 1 = 0$$

$$N_5 = \frac{1}{2} c (\xi - 1) (\xi + 1) (\eta - 1)$$

Applying boundary condition

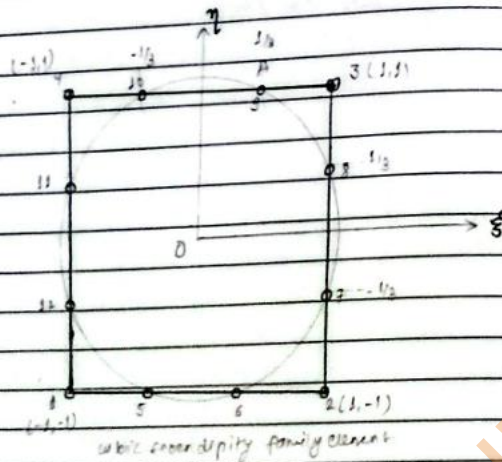
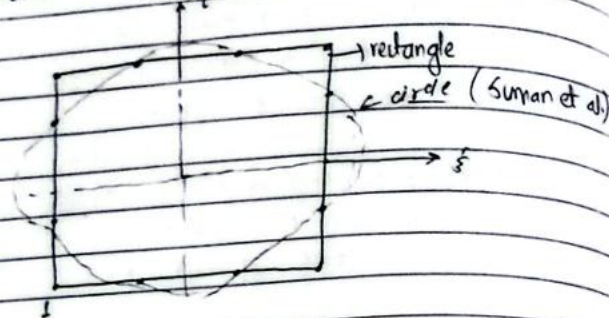
$$1 = c (0 - 1) (0 + 1) (-1 - 1)$$

$$\{c = \frac{1}{2}\}$$

$$N_5 = \frac{1}{2} c (\xi - 1) (\xi + 1) (\eta - 1)$$

(Shrivastava) → RBT

8-noded
Quadratic serendipity
family
12-noded Rectangle element
cubic serendipity family



→ $N_1 = 0$ for all nodes except 1 and 12 for node 1
 $N_1 = 0$ is satisfied for nodes 2, 7, 8, 3 if $1 - \xi = 0$ — (i)
 $N_1 = 0$ is satisfied for nodes 8, 9, 10, 4 if $1 - \eta = 0$ — (ii)

The points 5, 6, 7, 8, 9, 10, 11, 12 lie on circle.

$$\text{radius, } OA = \sqrt{\left(\frac{1}{3}\right)^2 + 1^2} = \frac{\sqrt{10}}{3}$$

At $(\frac{1}{3}, 1)$ & $(0, 0)$

Equation of circle:

$$\xi^2 + \eta^2 = \frac{10}{9}$$

$$\text{or, } \xi^2 + \eta^2 - \frac{10}{9} = 0 \quad \text{--- (iii)}$$

Satisfies $N_1 = 0$ for nodes 5 to 12,

So,

combining (i), (ii) & (iii)

$$N_1 = C(1 - \xi)(1 - \eta)\left(\xi^2 + \eta^2 - \frac{10}{9}\right)$$

→ satisfies for all nodes except node 1.

For node 1, $N_1 = 1$

$$\therefore 1 = C(1 - \xi_1)(1 - \eta_1)\left(\xi_1^2 + \eta_1^2 - \frac{10}{9}\right)$$

For, node 1, $\xi_1 = \eta_1 = -1$

$$\therefore C = \frac{9}{(2)(2)(9 + 9 - 10)} = \frac{9}{32}$$

Thus

$$N_1 = \frac{9}{32}(1 - \xi)(1 - \eta)\left(\xi^2 + \eta^2 - \frac{10}{9}\right)$$

Similarly,

$$N_2 = \frac{9}{32}(1 + \xi)(1 - \eta)\left(\xi^2 + \eta^2 - \frac{10}{9}\right)$$

$$N_3 = \frac{9}{32}(1 + \xi)(1 + \eta)\left(\xi^2 + \eta^2 - \frac{10}{9}\right)$$

$$N_4 = \frac{9}{32}(1 - \xi)(1 + \eta)\left(\xi^2 + \eta^2 - \frac{10}{9}\right)$$

For mid side node 5,

$1 - \xi = 0$ ensures, $N_1 = 0$ at node 2, 7, 8, 3

$1 - \eta = 0$ " " " " 3, 9, 10, 4

$1 + \xi = 0$ " " " " 4, 11, 12, 1

$1 + \eta = 0$ " " " " 12, 11, 10, 9

For mid side node 6,

$$N_5 = C(1 - \xi)(1 - \eta)(1 + \xi)(1 - 3\xi)$$

$$\text{At } \eta = 1 \rightarrow \xi = -\frac{1}{3}, \eta = 1 \text{ & } N_5 = 1 \Rightarrow C = \frac{9}{32}$$

Divide,

$$N_1 = \frac{9}{32} (1-\xi)(1-\eta)(1+\xi)(1-3\xi)$$

Sim

$$N_2 = \frac{9}{32} (1-\xi^2)(1-\eta)(1+3\xi)$$

$$N_3 = \frac{9}{32} (1-\eta^2)(1+\xi)(1+3\eta)$$

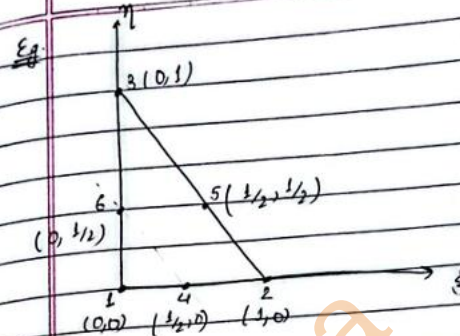
$$N_4 = \frac{9}{32} (1-\eta^2)(1+\xi)(1-3\eta)$$

$$N_5 = \frac{9}{32} (1-\xi^2)(1+\eta)(1+3\xi)$$

$$N_6 = \frac{9}{32} (1-\xi^2)(1+\eta)(1-3\xi)$$

$$N_7 = \frac{9}{32} (1-\eta^2)(1-\xi)(1+3\eta)$$

$$N_8 = \frac{9}{32} (1-\eta^2)(1-\xi)(1-3\eta)$$



→ Solⁿ:-

$$\begin{cases} N_1 = 1 & \text{at } \xi = 0, \eta = 0 \\ N_1 = 0 & \text{along } 1-5-3, \xi + \eta - 1 = 0 \\ N_1 = 0 & \text{along } 4-6, 2\xi + 2\eta - 1 = 0 \end{cases}$$

$$N_1 = C(\xi + \eta - 1)(2\xi + 2\eta - 1)$$

Applying Boundary condition.

$$1 = C(-1)(-1)$$

$$[C = 1]$$

$$\therefore N_1 = (\xi + \eta - 1)(2\xi + 2\eta - 1)$$

N₂

$$N_2 = 1 \text{ at } \xi = 1, \eta = 0$$

$$N_2 = 0 \text{ at } \xi = 1/2$$

$$N_2 = 0 \text{ at } \xi = 0$$

$$N_2 = C(\xi)(\xi - 1/2)$$

$$C = 2$$

$$N_2 = 2\xi(\xi - 1/2)$$

$$N_2 = \xi(2\xi - 1)$$

$$N_3 = \eta(2\eta - 1)$$

$$N_4 = 1 \text{ @ } \xi = 1/2, \eta = 0$$

$$N_4 = 0 \text{ @ } \xi = 0$$

$$N_4 = 0 \text{ @ } \xi + \eta - 1 = 0$$

$$N_4 = c\xi(\xi + \eta - 1)$$

$$c = -4$$

$$N_4 = 4\xi(1 - \xi - \eta)$$

$$N_5 = 1 \text{ @ } \xi = 1/2, \eta = 1/2$$

$$N_5 = 0 \text{ @ } \xi = 0, \eta = 0$$

$$N_5 = c\xi\eta$$

$$= 4\xi\eta$$

N₆

$$N_6 = 4\eta(1 - \xi - \eta)$$

$$N_6 = 1 \text{ @ } \xi = 0, \eta = 1/2$$

$$= 0 \text{ @ } \eta = 0$$

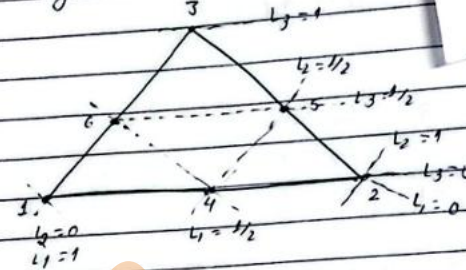
$$= 0 \text{ @ } \xi + \eta - 1 = 0$$

$$N_6 = c\eta(\xi + \eta - 1)$$

$$1 = c \times 1/2 (0 + 1/2 - 1)$$

$$c = -4$$

Ex. 6-Noded Triangle



→ Soln.

$$N_1 \Rightarrow$$

$$N_1 = 0 \Rightarrow L_1 = 1/2, L_2 = 0$$

$$N_1 = cL_1(L_1 - 1/2)$$

$$N_1 = 1, L_1 = 1$$

$$1 = c \times 1(1 - 1/2)$$

$$\Rightarrow [c = 2]$$

$$\therefore [N_1 = 2L_1(L_1 - 1/2)]$$

$$N_2 \Rightarrow$$

$$N_2 = 0 \text{ @ } L_2 = 1/2$$

$$\text{ @ } L_2 = 0$$

$$N_2 = cL_2(L_2 - 1/2)$$

$$N_2 = 1 \text{ @ } L_2 = 1$$

$$[c = 2]$$

$$\therefore N_2 = 2L_2(L_2 - 1/2)$$

Similarly,
 $N_3 = 2L_2 (L_2 - \frac{1}{2})$

$N_4 = 0$ @ $L_2 = 0$
 $L_1 = 0$

$N_4 = c L_1 L_2$

At boundary condition,

$N_4 = 1$ @ $L_1 = \frac{1}{2}, L_2 = \frac{1}{2}$

$[c = 4]$

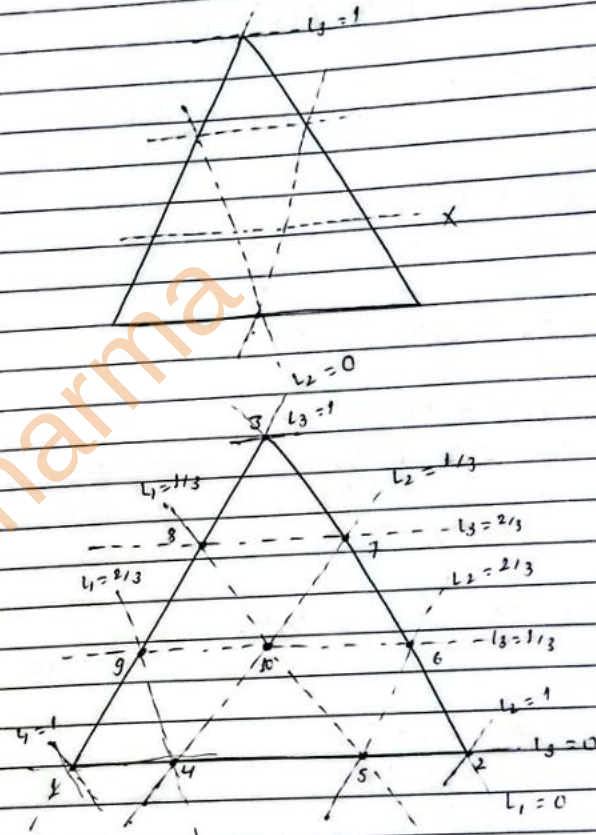
$[N_4 = 4L_1 L_2]$

Similarly,

$N_5 = 4L_2 L_3$

$N_6 = 4L_3 L_1$

10-Noded Triangle.



→ $N_1 \Rightarrow$

$N_1 = 0$ @ $L_1 = \frac{1}{3}$
 $L_2 = \frac{1}{3}$

$L_1 = 0$

$N_1 = c L_1 (L_1 - \frac{1}{3}) (L_1 - \frac{2}{3})$

Applying Boundary condition,

$N_1 = 1$ @ $L_1 = 1$

$[c = \frac{9}{2}]$ $[\therefore N_1 = \frac{9}{2} L_1 (L_1 - \frac{1}{3}) (L_1 - \frac{2}{3})]$

Similarly,

$$N_2 = \frac{9}{2} L_2 (L_2 - \frac{1}{3}) (L_2 - \frac{2}{3})$$

And, $N_3 = \frac{9}{2} L_3 (L_3 - \frac{1}{3}) (L_3 - \frac{2}{3})$

Now,

$$N_4 = 0 \text{ @ } L_2 = 0$$

$$L_1 = 0$$

$$L_3 = \frac{1}{3}$$

$$N_4 = C L_1 L_2 (L_1 - \frac{1}{3})$$

Applying BC

$$N_4 = 1 \text{ @ } L_1 = \frac{2}{3}$$

$$L_2 = \frac{1}{3}$$

$$C = \frac{27}{2}$$

$$N_4 = \frac{27}{2} L_1 L_2 (L_1 - \frac{1}{3})$$

$$N_5 = 0 \text{ @ } L_1 = 0$$

$$L_2 = \frac{1}{3}$$

$$L_3 = 0$$

$$N_5 = C L_1 L_2 (L_2 - \frac{1}{3})$$

Applying BC

$$N_5 = 1 \text{ @ } L_2 = \frac{2}{3}$$

$$L_1 = \frac{1}{3}$$

$$C = \frac{27}{2}$$

$$N_5 = \frac{27}{2} L_1 L_2 (L_2 - \frac{1}{3})$$

Similarly,

$$N_6 = \frac{27}{2} L_2 L_3 (L_2 - \frac{1}{3})$$

$$N_7 = \frac{27}{2} L_2 L_3 (L_3 - \frac{1}{3}) \quad \left\{ \begin{array}{l} -\frac{27}{2} L_2 L_3 (L_2 - \frac{2}{3}) \end{array} \right.$$

$$N_8 = \frac{27}{2} L_3 L_1 (L_3 - \frac{1}{3})$$

$$N_9 = \frac{27}{2} L_3 L_1 (L_1 - \frac{1}{3})$$

N_{10}

$$N_{10} = 0 \text{ @ } L_1 = 0$$

$$L_2 = 0$$

$$L_3 = 0$$

$$N_{10} = C L_1 L_2 L_3$$

Applying BC,

$$N_{10} = 1 \text{ @ } L_1 = \frac{1}{3}, L_2 = \frac{1}{3}, L_3 = \frac{1}{3}$$

$$C = 27$$

$$[N_{10} = 27 L_1 L_2 L_3]$$

Hermite Shape function (for Beam Element)

Let us consider a beam element in global coordinate system has a displacement & rotation u_1, u_2, u_3, u_4

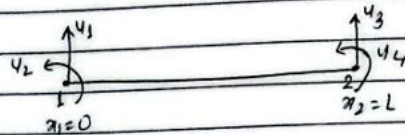
Let

F_1 = Shear force at node ①

M_1 = Bending Moment at node ①

F_2 = SF @ node ②

M_2 = BM @ node ②



At node ①, $x=x_1$

$$u(x) = u_1$$

$$u'(x) = \frac{du}{dx} = u_2$$

At node ②, $x=x_2=L$

$$u(x) = u_3$$

$$u'(x) = \frac{du}{dx} = u_4$$

Since there are 4 DOF, four parameters are selected

$$u = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$$

In matrix Form,

$$u = [1 \quad x \quad x^2 \quad x^3] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \quad \text{--- (1)}$$

$$\theta = \frac{du}{dx} = \alpha_2 + 2\alpha_3 x + 3\alpha_4 x^2 \quad \text{--- (2)}$$

At $x=0$, $u=u_1$, $\theta=u_2$

$$u_1 = \alpha_1$$

$$u_2 = \alpha_2$$

At $x=L$, $u=u_3$, $\theta=u_4$

$$u_3 = \alpha_1 + \alpha_2 L + \alpha_3 L^2 + \alpha_4 L^3$$

$$u_4 = \alpha_2 + 2\alpha_3 L + 3\alpha_4 L^2$$

In matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \frac{1}{L^4} \begin{bmatrix} L^4 & 0 & -3L^2 & 2L \\ 0 & L^4 & -2L^3 & L^2 \\ 0 & 0 & 3L^2 & -2L \\ 0 & 0 & -L^3 & L^2 \end{bmatrix}^T \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/L^2 & -2/L & 3/L^2 & -1/L \\ 2/L^3 & 1/L^2 & -1/L^2 & 1/L^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Now, substituting in eq (1)

$$u = [1 \quad x \quad x^2 \quad x^3] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

$$\theta = \frac{du}{dx}$$

$$u = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/L & -2/L & 3/L^2 & -1/L \\ 3/L^3 & 1/L^2 & -2/L^3 & 3/L^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Shape function

$$u = \left[\left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right) \left(\frac{x - \frac{2x^2}{L} + \frac{x^3}{L^2}\right) \left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right) \left(\frac{-x}{L} + \frac{3x^2}{L^2}\right) \right]$$

$$u = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Thus,

$$\Rightarrow [u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4]$$

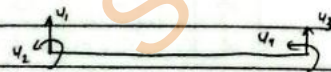
$$N_1 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}$$

$$N_2 = \frac{x - \frac{2x^2}{L} + \frac{x^3}{L^2}}{L}$$

$$N_3 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}$$

$$N_4 = \frac{-x}{L} + \frac{3x^2}{L^2}$$

← Hermite Shape function



$$\Rightarrow \theta = \frac{du}{dx}; \quad \begin{aligned} N_1' &= -\frac{6x}{L^2} + \frac{6x^2}{L^3} \\ N_2' &= 1 - \frac{4x}{L} + \frac{3x^2}{L^2} \\ N_3' &= \frac{6x}{L^2} - \frac{6x^2}{L^3} \\ N_4' &= -\frac{2x}{L} + \frac{3x^2}{L^2} \end{aligned}$$

* Graphical Representation of Hermite Shape functions

① For N_1



At node ①

$$x=0$$

$$N_1=1$$

$$N_1' = -\frac{6x}{L^2} + \frac{6x^2}{L^3}$$

$$= 0$$

node ②

$$x=L$$

$$N_1=0$$

$$N_1'=0$$

② For N_2 (related to rotation at node 1)

Node ①, $x=0$

$$N_2=0$$

$$N_2' = 1 - \frac{4x}{L} + \frac{3x^2}{L^2}$$

$$N_2' = 1$$

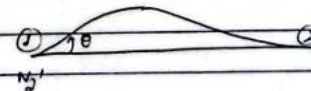
Node ②, $x=L$

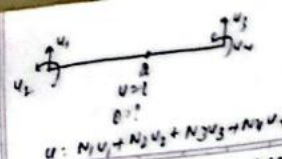
$$N_2=0$$

$$N_2'=0$$

$$u_2 = \theta = \frac{du}{dx} = \frac{d}{dx} (N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4)$$

$$u_2 = N_1' u_1 + N_2' u_2 + N_3' u_3 + N_4' u_4$$





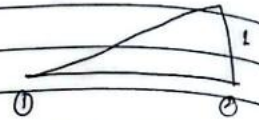
Equation for node 1

$$u = N_1 u_1 + N_2 u_2 + N_3 u_1' + N_4 u_2'$$

$$\theta = \frac{du}{dx} = N_1' u_1 + N_2' u_2 + N_3' u_1' + N_4' u_2'$$

(8) For N_3

Node 1	Node 2
$x=0$	$x=L$
$N_3=0$	$N_3=1$
$N_3'=0$	$N_3'=0$



(4) For N_4

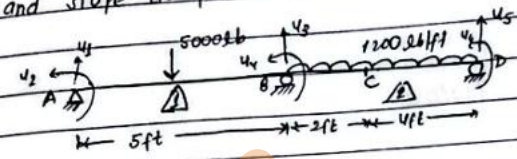
Node 1	Node 2
$x=0$	$x=L$
$N_4=0$	$N_4=0$
$N_4'=0$	$N_4'=1$



This is exactly the requirement of shape function hence first order Hermite functions are suitable as shape function

Hermite Shape Function → gives slope (along with displacement), particularly used for beam

(9) Determine nodal displacements, support reactions, Deflection and slope at point C



(Beam = Axial deformation not taken into account so AE term doesn't exist)

$$E = 30 \times 10^6 \text{ psi}$$

$$I = 305 \text{ in}^4$$

[6 x 6]

[FEM → only considers nodal displacement]
[No node ⇒ no info in FEM]

→ Solution:-

Step 1 Stiffness matrix

Element 1 → AB

$$[K_1] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$[5ft = 5 \times 12 = 60in]$$

$$= 4.236 \times 10^4 \begin{bmatrix} 12 & 360 & -720 & 360 \\ 360 & 24000 & -360 & 7200 \\ -720 & -360 & 7200 & -360 \\ 360 & 7200 & -360 & 14400 \end{bmatrix}$$

$$= 10^4 \begin{bmatrix} 50.83 & 1525 & -50.83 & 1525 \\ 1525 & 61000 & -1525 & 30500 \\ -50.83 & -1525 & 50.83 & -1525 \\ 1525 & 30500 & -1525 & 61000 \end{bmatrix}$$

8.7

1: $6 \times 12 = 72$ ft

$$[K_2] = 10^4 \begin{bmatrix} 2.54 & 1859 & -2.9 & 1059 & 0 & 0 \\ 1859 & 50833 & -1859 & 25416 & 0 & 0 \\ -2.9 & -1859 & 2.9 & 1059 & 0 & 0 \\ 1059 & 25416 & 1859 & 50833 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1059 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1059 \end{bmatrix}$$

Global stiffness matrix:

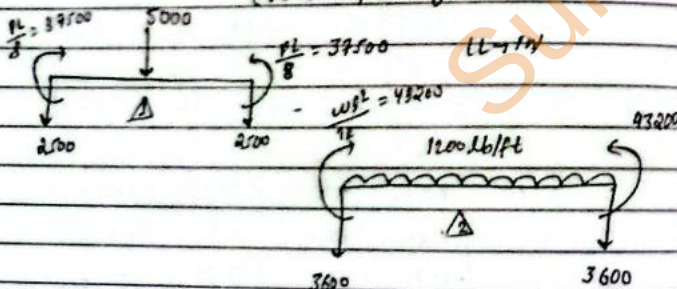
$$[K] = 10^4 \begin{bmatrix} 50833 & 1859 & -1083 & 1725 & 0 & 0 & 1 \\ 1859 & 61000 & -1525 & 30700 & 0 & 0 & 2 \\ -1083 & -1525 & 8082 & -966 & -294 & 1059 & 3 \\ 1725 & 30700 & -966 & 111833 & -1059 & 25416 & 4 \\ 0 & 0 & -294 & 1059 & 294 & 1059 & 5 \\ 0 & 0 & 1059 & 25416 & 1059 & 50833 & 6 \end{bmatrix}$$

Step 2: Force Matrix

External loads:

(Nodal loads matrix)

(Distribute/Resolving in nodal)



Nodal force matrix

$$[F_1] = \begin{bmatrix} -2500 \\ -37500 \\ -3600 \\ 43200 \end{bmatrix}$$

$$[F_2] = \begin{bmatrix} -3600 \\ -43200 \\ -3600 \\ 43200 \end{bmatrix}$$

$$[F] = \begin{bmatrix} -2500 + R_A \\ -37500 + 0 \\ (-3600) + R_B \\ (37500 - 43200) + 0 \\ -3600 + R_D \\ 43200 + 0 \end{bmatrix} = \begin{bmatrix} -2500 + R_A \\ -37500 + 0 \\ -3600 + R_B \\ -5700 \\ -3600 + R_D \\ 43200 \end{bmatrix}$$

Step 3: Displacement matrix

B.C

(Support settle of the beam)
 $u_1, u_3, u_5 = 0$

$$\{u\} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} 0 \\ u_2 \\ 0 \\ u_4 \\ 0 \\ u_6 \end{bmatrix}$$

Step ④ Apply Boundary conditions
(Reduced matrix is: (1, 3, 5 → 11 in 0))

$$[K] \{u\} = \{F\}$$

$$10^4 \begin{bmatrix} 6100 & 3050 & 0 \\ 3050 & 11133 & 2541 \\ 0 & 2541 & 5083 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_3 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} -37500 \\ -5700 \\ 43200 \end{Bmatrix}$$

On solving

$$\begin{Bmatrix} u_1 \\ u_3 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} -5 \times 10^{-5} \\ -1 \times 10^{-5} \\ 9 \times 10^{-5} \end{Bmatrix} \text{ rad}$$

Step ⑤

Support reactions: (R_A, R_B, R_D)

Step ⑥ Displacement and slope at pt. C.

($a = 24 \text{ ft} = 24 \text{ in}$)

Displacement

$$u = H_1 u_3 + H_2 u_4 + H_3 u_5 + H_4 u_6 \quad (H' \text{ or } N)$$

$$= H_2 u_4 + H_4 u_6$$

$$N_2 = H_2 = \left[a - \frac{2a^2}{L} + \frac{3a^3}{L^2} \right] = 10.67 \quad (\text{in in})$$

$$N_4 = H_4 = \left[-\frac{2a^2}{L} + \frac{3a^3}{L^2} \right] = -5.33 \quad (\text{in in})$$

$$u = 10.67 \times (-1 \times 10^{-5}) - 5.33 \times (9 \times 10^{-5}) = -5.38 \times 10^{-4} \text{ in (displacement at C)}$$

Rotation/Slope

$$\theta = \frac{du}{da} = \frac{d}{da} (H_2 u_4 + H_4 u_6) = H_2' u_4 + H_4' u_6$$

$$H_2' = \frac{d(H_2)}{da} = \left[1 - \frac{4a}{L} + \frac{3a^2}{L^2} \right] = 0$$

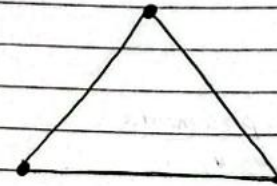
$$H_4' = \frac{d(H_4)}{da} = \left[-\frac{2a}{L} + \frac{3a^2}{L^2} \right] = -0.333$$

$$\theta = 0 \times (-1 \times 10^{-5}) - 0.333 \times (9 \times 10^{-5}) = -3 \times 10^{-5} \text{ rad (slope at C)}$$

Constant Strain Triangle

- 3-noded triangular element
- was one of the first elements extensively used for continuum stress analysis.
- Use of areas whose strain gradient are small.
- Use of mesh transitions areas (fine to coarse mesh)
- Not recommended for general analysis as a very large number of these elements are required for reasonable accuracy.
- CST use linear shape functions i.e. the solution within the system element is approximated as a linear function. This leads to simple & efficient computation of element stiffness matrices & load vectors.
- Consider displacement equation for this element

$$u = a + bx$$



$$\rightarrow \text{strain} = \frac{du}{dx} = b (\text{constant})$$

- The original element will either magnify with constant strain at all places or zoom out. Take quadratically formulated triangle (6 nodes)

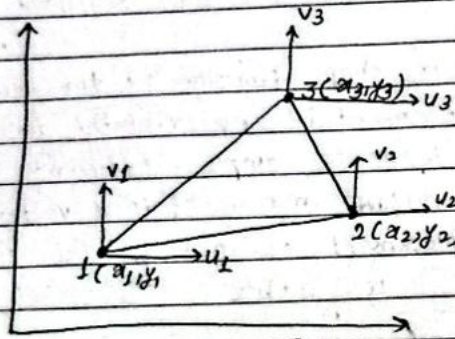


$$u = a + bx + cx^2$$

$$\rightarrow \text{strain} = b + 2cx \rightarrow \text{linear variation}$$

→ CST is not recommended to be used in the critical area of the structure such as stress concentration, edge of the holes, corners, etc

Shape function: for CST using polynomial functions:-



→ Solⁿ:

There are 6 nodal displacements

$u_1, v_1, u_2, v_2, u_3, v_3$

$$\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y \\ v &= \alpha_4 + \alpha_5 x + \alpha_6 y \end{aligned} \quad \text{--- (1)}$$

Apply Boundary conditions:

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$

$$u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$

$$u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

$$\text{i.e. } \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}^T \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\text{where, } 2A = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}^T \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Now equation (1) becomes,

$$u = [1 \quad x \quad y] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$= [1 \quad x \quad y] \times \frac{1}{2A} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\therefore u = \begin{bmatrix} \frac{a_1 + b_1 x + c_1 y}{2A} & \frac{a_2 + b_2 x + c_2 y}{2A} & \frac{a_3 + b_3 x + c_3 y}{2A} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

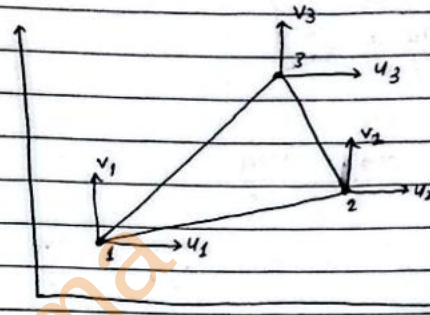
$$u = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

Similarly,

$$v = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

Strain - Displacement relationship for CST.



At any point in CST element used for plane stress or plane strain problems, there are three strain components $\epsilon_x, \epsilon_y, \gamma_{xy}$.

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} \quad \text{--- (1)}$$

Displacement at any point is given by,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} \quad \text{--- (11)}$$

$$\left\{ N_1 = \frac{q_1 + b_1 x + c_1 y}{2A} \Rightarrow \frac{\partial N_1}{\partial x} = \frac{b_1}{2A} \right\} \frac{\partial N_1}{\partial y} = \frac{c_1}{2A}$$

$$\therefore \{E\} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$\epsilon_x = \frac{\partial u}{\partial x}$$

$$= \frac{1}{2A} [N_1 u_1 + N_2 u_2 + N_3 u_3]$$

$$= \frac{1}{2A} [b_1 u_1 + b_2 u_2 + b_3 u_3]$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{1}{2A} [N_1 v_1 + N_2 v_2 + N_3 v_3]$$

$$= \frac{1}{2A} [c_1 v_1 + c_2 v_2 + c_3 v_3]$$

$$\gamma_{xy} = \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}$$

$$= \frac{1}{2A} [b_1 v_1 + b_2 v_2 + b_3 v_3 + c_1 u_1 + c_2 u_2 + c_3 u_3]$$

In matrix form,

$$\{E\} = \frac{1}{2A} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\{E\} = [B] \{u\} \quad \text{--- (*)}$$

This is strain-displacement relation:-

$$\begin{array}{l|l} b_1 = x_2 - x_3 = y_{23} & c_1 = x_3 - x_1 = x_{31} \\ b_2 = x_3 - x_1 = x_{31} & c_2 = x_1 - x_3 = x_{13} \\ b_3 = x_1 - x_2 = x_{12} & c_3 = x_2 - x_1 = x_{21} \end{array}$$

Stiffness matrix for CST

Using Energy concept

~~External and Internal en~~

External energy = Internal energy

$$\Rightarrow \frac{1}{2} \{u\}^T \{F\} = \frac{1}{2} \iiint_V \{E\}^T \{F\} dv$$

$$\text{or, } \{u\}^T \{F\} = \iiint_V \{u\}^T [B]^T [D] \{E\} dv$$

$$\text{or, } \{u\}^T \{F\} = \iiint_V \{u\}^T [B]^T [D] [B] \{u\} dv$$

$$= \{u\}^T [B]^T [D] [B] \{u\} \iiint_V dv$$

$$= A \cdot t \{u\}^T [B]^T [D] [B] \{u\}$$

or $\{u\}^T \{F\} = A \cdot t \cdot \{u\}^T [B]^T [D] [B] \{u\} = 0$

or $\{u\}^T [\{F\} - A \cdot t \cdot [B]^T [D] [B] \{u\}] = 0$

Since, $\{u\}^T \neq 0$

$\{F\} = A \cdot t \cdot [B]^T [D] [B] \{u\}$

$\{F\} = [K] \{u\} \quad \text{--- (x)}$

Here,

$[K] = A \cdot t \cdot [B]^T [D] [B] \quad \text{--- (xx)}$

CST Force.

2070 Bhadra.

$$E = 30 \times 10^6 \text{ psi}$$

$$t = 0.3 \text{ inch}$$

$$\gamma_b = 460 \text{ lb/in}^3$$

$$\nu = 0.3$$

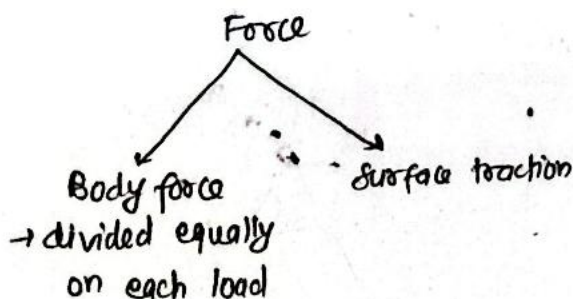
$$T_3 = 360 \text{ psi (surface traction)}$$

Given: It is plane stress

Find Nodal disp. & stress.

Solution:

Step 1: Modeling



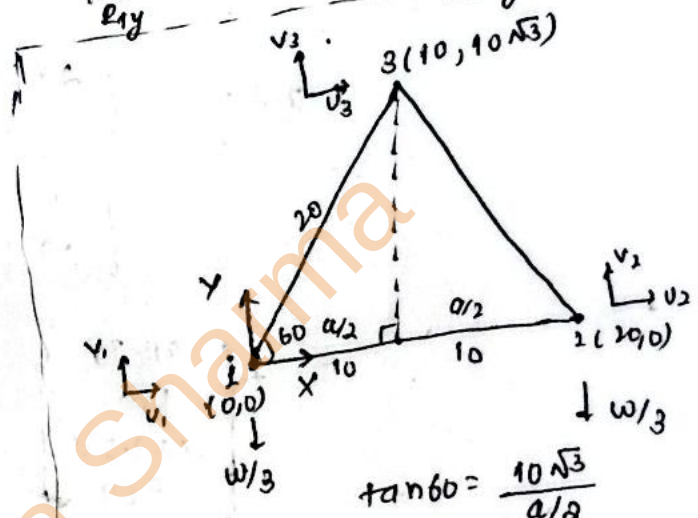
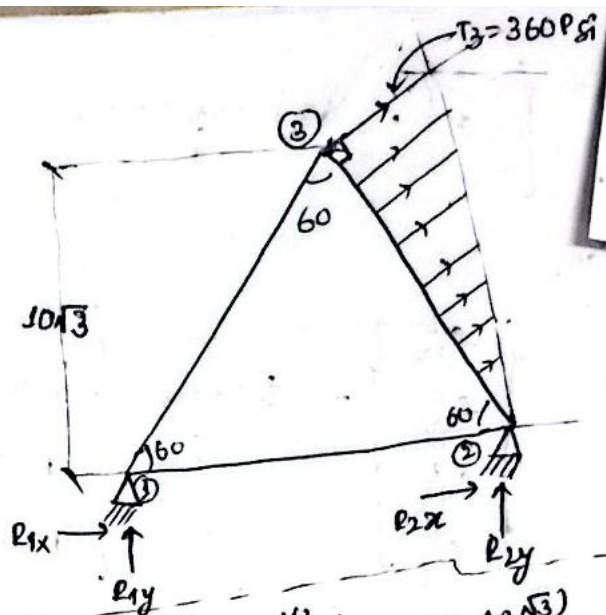
Step 2: Force matrix

$$\gamma_b = 460 \text{ lb/in}$$

$$t = 0.3 \text{ in}$$

$$\text{Body force} = \begin{Bmatrix} f_{y1}^b \\ f_{y2}^b \\ f_{y3}^b \end{Bmatrix} = \frac{\gamma_b A t}{3} \begin{Bmatrix} -1 \\ -1 \\ -1 \end{Bmatrix}$$

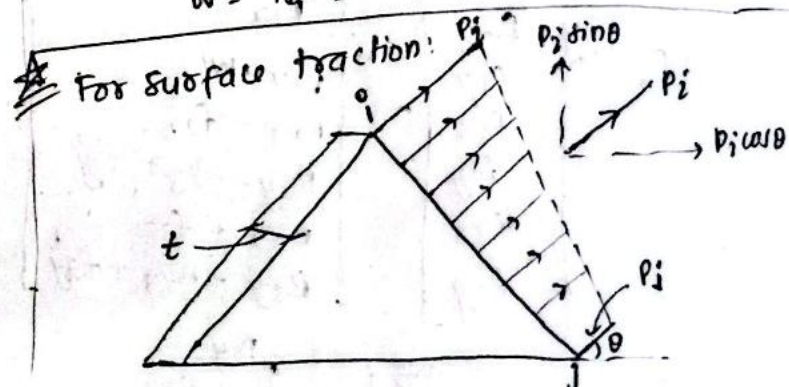
$$A = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 100\sqrt{3} \text{ in}^2$$



$$\tan 60 = \frac{10\sqrt{3}}{a/2}$$

$$[\Rightarrow a = 20 \text{ in}]$$

$$W = \gamma_b \times V = \gamma_b \times A \times t$$



$$\begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{Bmatrix} = \frac{t \times L_{ij}}{6} \begin{Bmatrix} 2P_1 \cos \theta + P_2 \cos \theta \\ 2P_1 \sin \theta + P_2 \sin \theta \\ P_1 \cos \theta + 2P_2 \cos \theta \\ P_1 \sin \theta + 2P_2 \sin \theta \end{Bmatrix}$$

Side 3-1 → Cartesian
Side 3-2 → load; f₃₋₁ & f₃₋₂ multiply

$$\{f_b\} = \begin{Bmatrix} -1 \\ -1 \\ -1 \end{Bmatrix} \cdot \frac{460 \times 100\sqrt{3} \times 0.3}{3} = 7967.434 \begin{Bmatrix} -1 \\ -1 \\ -1 \end{Bmatrix}$$

To action,

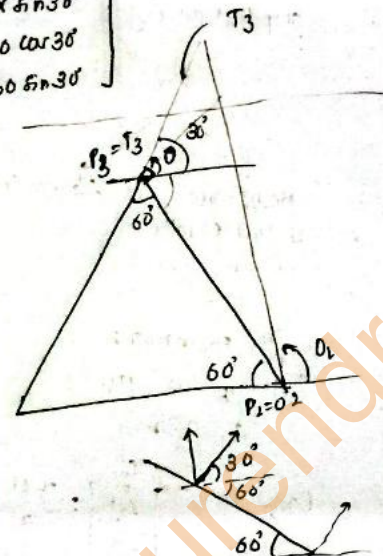
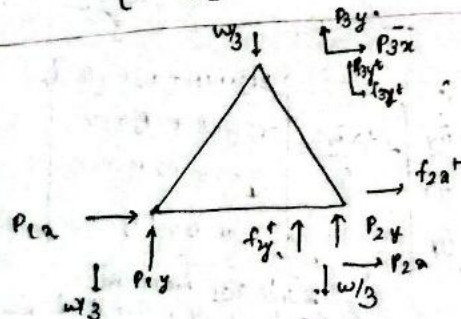
$$\{f_t\} = \begin{Bmatrix} f_{1x}^t \\ f_{1y}^t \\ f_{2x}^t \\ f_{2y}^t \\ f_{3x}^t \\ f_{3y}^t \end{Bmatrix} = \frac{t \times L_{2-3}}{6} \begin{Bmatrix} 2P_2 \cos \theta + P_3 \cos \theta \\ 2P_2 \sin \theta + P_3 \sin \theta \\ P_2 \cos \theta + 2P_3 \cos \theta \\ P_2 \sin \theta + 2P_3 \sin \theta \end{Bmatrix}$$

$$= \frac{0.3 \times 20}{6} \begin{Bmatrix} 2 \times 0 + 360 \times \cos 30^\circ \\ 0 + 360 \times \sin 30^\circ \\ 0 + 2 \times 360 \cos 30^\circ \\ 0 + 2 \times 360 \sin 30^\circ \end{Bmatrix}$$

$$= \begin{Bmatrix} 311.77 \\ 180 \\ 623.54 \\ 360 \end{Bmatrix}$$

$\therefore$ Force vectors

$$\{F\} = \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \begin{Bmatrix} P_{1x} \\ P_{1y} - w/3 \\ P_{2x} + f_{1x}^t \\ P_{2y} - w/3 + f_{1y}^t \\ P_{3x} + f_{2x}^t \\ P_{3y} - w/3 + f_{2y}^t \end{Bmatrix}$$



$\theta = \text{Angle with } x\text{-axis}$
(anti-clockwise)

$$\{F\} = \begin{Bmatrix} P_{1x} \\ P_{1y} - 7967.434 \\ P_{2x} + 311.77 \\ P_{2y} - 7967.434 + 180 \\ 0.623.54 \\ -7967.434 + 360 \end{Bmatrix} = \begin{Bmatrix} P_{1x} \\ P_{1y} - 7967.434 \\ P_{2x} + 311.77 \\ P_{2y} - 7787.434 \\ 623.54 \\ -7607.434 \end{Bmatrix}$$

Step 3 Displacement vector

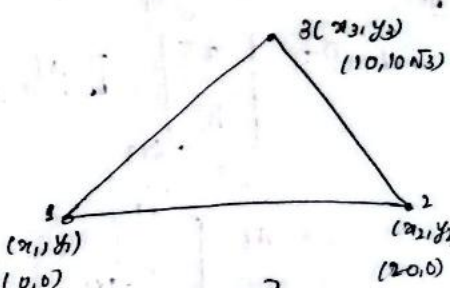
$$\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \end{Bmatrix}$$

Since there is hinge support @ node 2 & 3.

Step 4: Stiffness matrix:

$$[K] = A [B] [B]^T [D] t$$

$$6 \times 6 = A + [B]^T [D] [B]$$



$$[B] = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$[B] = \frac{1}{2A} \begin{bmatrix} -10\sqrt{3} & 0 & 10\sqrt{3} & 0 & 0 & 0 \\ 0 & -10 & 0 & -10 & 0 & 20 \\ -10 & -10\sqrt{3} & -10 & 10\sqrt{3} & 20 & 0 \end{bmatrix} 3 \times 6$$

$$[B]^T = \frac{1}{2A} \begin{bmatrix} -10\sqrt{3} & 0 & -10 \\ 0 & -10 & -10\sqrt{3} \\ 10\sqrt{3} & 0 & -10 \\ 0 & -10 & 10\sqrt{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} a^T \\ b^T \end{matrix} 6 \times 3$$

[D] = Plane stress condition.

for Inplane loading

$$[D]_{3 \times 3} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\nu}{2}\right) \end{bmatrix}$$

$$= \frac{30 \times 10^6}{1-0.3^2} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$= 32.967 \times 10^6 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}_{3 \times 3}$$

Now,

$$[k] = A t \begin{bmatrix} a^T \\ b^T \end{bmatrix}_{6 \times 3} [D]_{3 \times 3} \begin{bmatrix} a \\ b \end{bmatrix}_{3 \times 6}$$

$$= A t \begin{bmatrix} a^T \\ b^T \end{bmatrix}_{2 \times 1} [D]_{3 \times 1} \begin{bmatrix} a \\ b \end{bmatrix}_{1 \times 1}$$

$$= A t \begin{bmatrix} a^T \\ b^T \end{bmatrix}_{2 \times 1} [ca \quad cb]_{1 \times 2}$$

$$[k] = A t \begin{bmatrix} a^T c a & a^T c b \\ b^T c a & b^T c b \end{bmatrix}_{2 \times 2}$$

$$[k] = \frac{32.967 \times 10^6 \times A t}{4 \text{ ft}^2} \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \\ 335 & 112.58 & -265 & -8.66 & -103.92 & -70 \\ 112.58 & 205 & 8.66 & -5 & -121.2 & -200 \\ -265 & 8.66 & 335 & 112.5 & -70 & 103.92 \\ -8.66 & -5 & 112.5 & 205 & 121.2 & -200 \\ -103.92 & -70 & 121.2 & -200 & 140 & 0 \\ -70 & -200 & 103.92 & 0 & 0 & 400 \end{bmatrix}$$

Step 5

$$\{F\} = [k] \{u\}$$

Applying B.C.

$$u_1 = v_1 = u_2 = v_2 = 0$$

$$[k] = \frac{1.713 \times 10^9}{4 \text{ ft}^2} \begin{bmatrix} 140 & 0 \\ 0 & 400 \end{bmatrix}$$

$$\therefore \begin{Bmatrix} 62354 \\ -7607.434 \end{Bmatrix} = \left(\frac{1.713 \times 10^9}{4 \text{ ft}^2} \right) \begin{bmatrix} 140 & 0 \\ 0 & 400 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix}$$

$$\begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0.000312 \\ -0.00133 \end{Bmatrix} \text{ in}$$

34 Element stress

A $\{\sigma\} = [D]\{\epsilon\}$
 $= [D][B]\{d\}$

$$\{\sigma\} = \frac{32.967 \times 10^6}{2A} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \begin{bmatrix} 10A \\ 0 \\ -10 \end{bmatrix} \begin{matrix} (a) \\ (b) \\ (c) \end{matrix}$$

$$\begin{bmatrix} 10A & 0 & 10A \\ 0 & -10 & -10 \\ -10 & -10A & 20 \end{bmatrix} \begin{matrix} (d) \\ (e) \\ (f) \end{matrix}$$

$$[a]_m [b \ c] \begin{Bmatrix} d \\ e \end{Bmatrix}$$

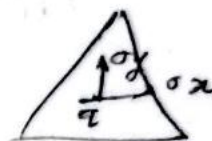
$$= [a \ b \ a \ c] \begin{Bmatrix} d \\ e \end{Bmatrix}$$

$$= [a \ b \ d \ a \ c \ e]$$

Thus

$$\{\sigma\} = \frac{32.967 \times 10^6}{2A} \begin{bmatrix} -3 & 0 & 6 \\ -10 & 0 & 26 \\ 1.0621 & 7 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ y_3 \\ y_3 \end{Bmatrix} \begin{matrix} 0.000312 \\ -0.00125 \end{matrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -7594.37 \\ -25314.56 \\ 2078.45 \end{Bmatrix}$$



* If only nodal displacement & stress is asked
 we reduced matrix $[k]$

$$[k] = A + [B]^T [D] [B]$$

$2 \times 3 \quad 3 \times 3 \quad 8 \times 1$

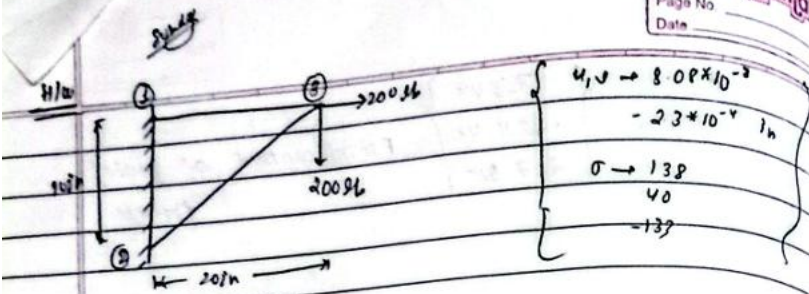
$$B = \begin{bmatrix} y_{23} & 0 & y_{31} \\ 0 & x_{32} & 0 \\ x_{32} & y_{23} & x_{23} \end{bmatrix} \begin{matrix} 6 \\ x_{23} \\ y_{31} \end{matrix} \begin{bmatrix} y_{12} & 0 \\ 0 & x_{21} \\ x_{12} & y_{12} \end{bmatrix}$$

$$[\quad]_{2 \times 2}$$

$$B^T = \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{23} \\ 0 & x_{23} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{matrix} u_3 \\ v_3 \end{matrix}$$

* If reaction asked

find full $[K]_{6 \times 6}$



$t = 0.3 \text{ in}$
 $\nu = 0.3$
 $E = 30 \times 10^6 \text{ psi}$

Find Nodal displacements & stresses.

$[k] = tA [B]^T [D] [B]$

Considering plane stress

(a) $[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$
 $32.967 \times 10^6 \text{ psi}$

$= 32.967 \times 10^6 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$

(b) $[B] = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$

$A = \frac{1}{2} \times 10 \times 20 = 100 \text{ in}^2$

$[B] = \frac{1}{200} \begin{bmatrix} -10 & 0 & 0 & 0 & 10 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & -10 & -20 & 0 & 0 & 10 \end{bmatrix}$

(a) (b)

$[k] = tA \times \frac{1}{20} \times \frac{1}{20} \times 32.967 \times 10^6 \begin{bmatrix} a^T \\ b^T \end{bmatrix} [c] \begin{bmatrix} a \\ b \end{bmatrix}$

$= \frac{1}{40} \times 32.967 \times 10^6 \begin{bmatrix} a^T c a & a^T c b \\ b^T c a & b^T c b \end{bmatrix}$

$24.725 \times 10^3 \text{ lb/in}$

$= 24.725 \times 10^3 \begin{bmatrix} 240 & -130 & -140 & 60 & -100 & 70 \\ -130 & 435 & 70 & -400 & 60 & -35 \\ -140 & 70 & 140 & 0 & 0 & -70 \\ 60 & -400 & 0 & 400 & -60 & 0 \\ -100 & 60 & 0 & -60 & 100 & 0 \\ 70 & -35 & -70 & 0 & 0 & 35 \end{bmatrix} \text{ lb/in.}$

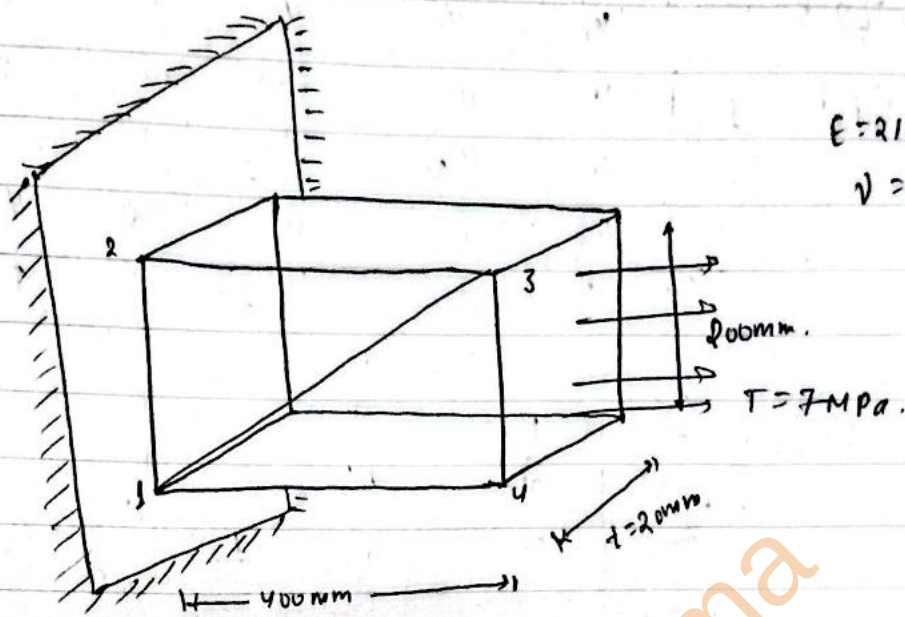
$\{F\} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \end{Bmatrix} ; \{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$

$u_3 = 8.09 \times 10^{-5} \text{ in}$
 $u_4 = -2.31 \times 10^{-4} \text{ in}$

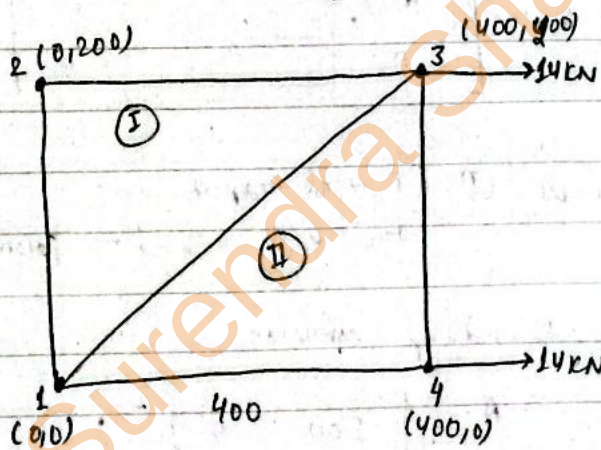
$\{\sigma\} = [D] \{\epsilon\} = [D] [B] \{u\}$

$\{\sigma\} = \frac{32.967 \times 10^6}{20 \times 100} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 0 \\ 0 & 10 \end{bmatrix} \begin{Bmatrix} 8.09 \times 10^{-5} \\ -2.31 \times 10^{-4} \end{Bmatrix}$

$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 133.35 \\ 40 \\ -133.27 \end{Bmatrix} \text{ psi}$



Find nodal displacements & element stresses.
 (Strut not given so neglect body forces)



Step 2) stiffness matrix:

Element I

$$[K] = t \cdot A \cdot [B]^T [D] [B]$$

$[B] = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$

local numbers $\rightarrow$ $u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3$

Global numbers

Global

$$[B] = \frac{1}{2A} \begin{bmatrix} u_1 & v_1 & u_3 & v_3 & u_2 & v_2 \\ 0 & 0 & 200 & 0 & -200 & 0 \\ 0 & -400 & 0 & 0 & 0 & 400 \\ -400 & 0 & 0 & 200 & 400 & -200 \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \quad \quad \uparrow \quad \uparrow$

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$= \frac{210 \times 10^6}{1-0.3^2} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

Global nodes: ① & ④ → fix support

u_1, v_1 & $u_4, v_4 = 0$ (eliminating these rows in $[B]$)

Applying boundary condition,

$$[k] = t \cdot A \cdot \begin{bmatrix} 200 & 0 \\ 0 & 0 \\ 0 & 400 \end{bmatrix}^T \begin{bmatrix} [D] & [B] \end{bmatrix}$$

calculator

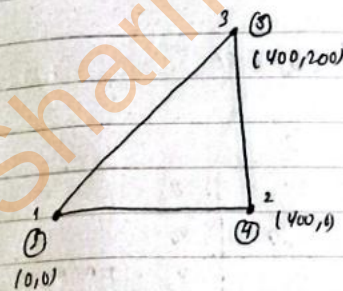
$$= t \cdot A \cdot \begin{bmatrix} 200 & 0 \\ 0 & 0 \\ 0 & 400 \end{bmatrix}^T \frac{210 \times 10^6}{1-0.3^2} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= 800 \times 10^6 \begin{bmatrix} 9.23 & 0 \\ 0 & 3.23 \end{bmatrix} \begin{bmatrix} u_3 \\ v_3 \end{bmatrix} \text{ kN/mm}$$

$$= 28846.15 \begin{bmatrix} 40000 & 0 \\ 0 & 14000 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ kN/mm}$$

Element ②

$$= 28846.15 \times 10^6 \begin{bmatrix} 40 & 0 \\ 0 & 14 \end{bmatrix} \text{ kN/mm}$$



$$[B] = \frac{1}{2A} \begin{bmatrix} u_1 & v_1 & u_4 & v_4 & u_3 & v_3 \\ -200 & 0 & 200 & 0 & 0 & 0 \\ 0 & 0 & 0 & 400 & 0 & 400 \\ 0 & -200 & 400 & 200 & 400 & 0 \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \quad \quad \uparrow \quad \uparrow$

$$[D] = \frac{210 \times 10^6}{1-0.3^2} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$u_1, v_1 = 0$

(12/12/21/3/4)

(4/4)

$$[K] = t \cdot A \cdot [B]^T [D] [B]$$

$$A = \frac{1}{2} \times 400 \times 200 \text{ mm}^2 \quad t = A = \frac{1}{2A} \begin{bmatrix} 200 & 0 & 0 & 0 \\ 0 & 400 & 0 & 400 \\ 400 & 200 & 400 & 0 \end{bmatrix}^T \times \frac{210 \times 10^6}{1-0.3^2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{2A} \begin{bmatrix} 200 & 0 & 0 & 0 \\ 0 & 400 & 0 & 400 \\ 400 & 200 & 400 & 0 \end{bmatrix}$$

$$= \frac{1}{2} \cdot A \cdot \frac{E}{1-0.3^2} \cdot \frac{1}{4A^2} [B]^T [D] [B]$$

$$= 2885 \times 10^6 \begin{bmatrix} 96 & -52 & -56 & 24 \\ -52 & 174 & 28 & -160 \\ -56 & 28 & 96 & 0 \\ 24 & -160 & 0 & 174 \end{bmatrix} \begin{matrix} u_4 \\ v_4 \\ u_3 \\ v_3 \end{matrix} \text{ FN/mm}$$

Step 2

Global Stiffness matrix:

$$[K] = 2885 \times 10^6 \begin{bmatrix} 96 & -52 & -56 & 24 \\ -52 & 174 & 28 & -160 \\ -56 & 28 & 96 & 0 \\ 24 & -160 & 0 & 174 \end{bmatrix} \begin{matrix} u_4 \\ v_4 \\ u_3 \\ v_3 \end{matrix}$$

Force Matrix

$$\begin{Bmatrix} F_{4x} \\ F_{4y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{Bmatrix} 14 \\ 0 \\ 14 \\ 0 \end{Bmatrix} \text{ kN}$$

Displacement matrix:

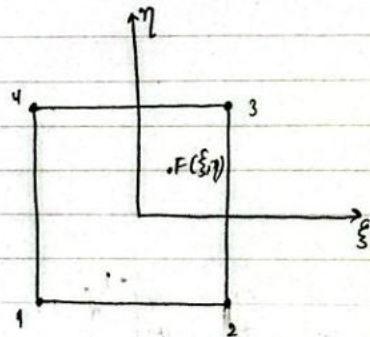
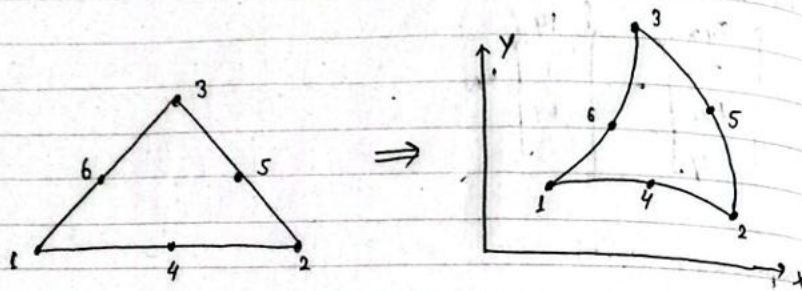
$$\{u\} = \begin{Bmatrix} u_4 \\ v_4 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\{F\} = [K] \{u\}$$

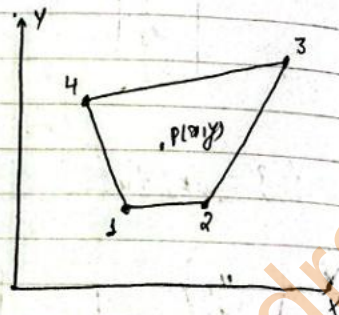
$$\begin{Bmatrix} u_4 \\ v_4 \\ u_3 \\ v_3 \end{Bmatrix} = \frac{1}{2885 \times 10^6} \begin{bmatrix} 96 & -52 & -56 & 24 \\ -52 & 174 & 28 & -160 \\ -56 & 28 & 96 & 0 \\ 24 & -160 & 0 & 174 \end{bmatrix}^{-1} \begin{Bmatrix} 14 \\ 0 \\ 14 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1.3 \times 10^{-8} \\ 2 \times 10^{-9} \\ 1.2 \times 10^{-8} \\ 8 \times 10^{-10} \end{Bmatrix} \text{ mm}$$

Isoparametric formulation:-



Parent element in natural coordinate system



Mapped element in Global co-ordinate system.

The various elements we have seen have straight edges, To take care of curved boundaries, isoparametric concept is used in properly mapping the curved boundaries. The concept of mapping changes regular triangle and rectangular elements in natural coordinate system to arbitrary shapes in Global coordinate system.

isoparametric, Superparametric & subparametric elements.

If same shape function are used to define geometry and displacement in an element, the element is called isoparametric element.

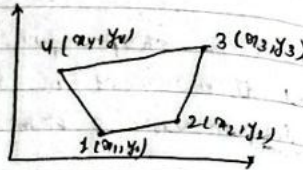
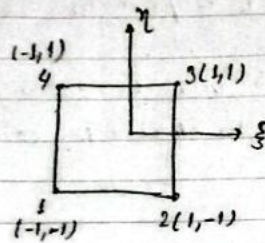
If more number of nodes are used to define the geometry of elements, then, in defining displacement, the element is called 'superparametric element'.

If more number of nodes are used to define displacement, then, in defining geometry, the element is called subparametric element.

Coordinate Transformation

Like shape function used for defining displacement of any point inside the element, shape functions are also used for coordinate transformation from natural coordinate to global cartesian system.

$$\begin{aligned}x &= N_1 x_1 + N_2 x_2 + \dots + N_n x_n \\y &= N_1 y_1 + N_2 y_2 + \dots + N_n y_n \\z &= N_1 z_1 + N_2 z_2 + \dots + N_n z_n\end{aligned}$$



Q10: Determine the cartesian coordinate of point.
 $\xi = 0.5$, $\eta = 0.6$

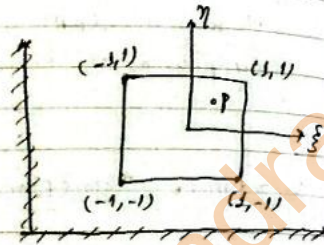
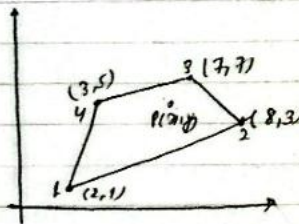


fig: Mapped element

$$N_1 = \frac{(1-\xi)(1-\eta)}{4} = 0.05$$

$$N_2 = \frac{(1+\xi)(1-\eta)}{4} = 0.15$$

$$N_3 = \frac{(1+\xi)(1+\eta)}{4} = 0.6$$

$$N_4 = \frac{(1-\xi)(1+\eta)}{4} = 0.2$$

Conjugate-Gradient Method:

If for a system of linear equations $Ax = b$, the coefficient matrix A is symmetric ($A^T = A$) & positive definite ($Z^T A Z > 0$), the equation can be solved by conjugate gradient method.

Iterative method of high convergence.

Steps/Algorithm:-

① Initially, for $i=0$, start with initial value
 $\{x\}_i = \{x\}_0$ ($= \{0\}$ if not given)

② Evaluate initial residual value
 $\{r\}_0 = \{b\} - [A]\{x\}_0$

③ Assign initial search direction:
 $\{v\}_i = \{v\}_0 = \{r\}_0$

④ Evaluate scalar α_i

$$\alpha_i = \frac{\{r\}_i^T \{r\}_i}{\{v\}_i^T [A] \{v\}_i}$$

here,
 $\{z\}_i = [A]\{v\}_i$

⑤ Evaluate
 $\{x\}_{i+1} = \{x\}_i + \alpha_i \{v\}_i$

$$\{r\}_{i+1} = \{r\}_i - \alpha_i \{z\}_i$$

⑥ Stop if converged i.e.
 $\{r\}_{i+1}$ is within permissible range then goto step ①
 otherwise goto step ⑦

⑦ Evaluate scalar β_i

$$\beta_i = \frac{\{x\}_i^T \{x\}_{i+1}}{\{x\}_i^T \{x\}_i}$$

⑧ Evaluate new search direction

$$\{x\}_{i+1} = \{x\}_{i+1} + \beta_i \{x\}_i$$

⑨ Assign $i=i+1$

⑩ goto step ⑧

⑪ Solution is $\{x\}_{i+1}$

⑫ End.

Eg:

$$3x_1 - 4x_2 = 2$$

$$4x_1 - x_2 = 9$$

Soln:

System of linear equation in matrix form is

$$\begin{bmatrix} 3 & -4 \\ 4 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 9 \end{Bmatrix}$$

Here coefficient matrix is not symmetric.
 Multiply the first eqⁿ by -1 to make coefficient matrix symmetric.

$$\begin{bmatrix} -3 & 4 \\ 4 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 9 \end{Bmatrix}$$

$$[A] \{x\} = \{b\}$$

Iteration 0 (initial sol)

Step 1: Assign $\{x\}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

Step 2: $\{x\}_0 = \{b\} - [A] \{x\}_0$
 $= \begin{Bmatrix} -2 \\ 9 \end{Bmatrix} - \begin{bmatrix} -3 & 4 \\ 4 & -1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$$= \begin{Bmatrix} -2 \\ 9 \end{Bmatrix}$$

Step 3: $\{x\}_1 = \{x\}_0 = \begin{Bmatrix} -2 \\ 9 \end{Bmatrix}$

$$\alpha_1 = \frac{\{x\}_1^T \{x\}_1}{\{x\}_1^T \{x\}_1}$$

$$\{x\}_2 = [A] \{x\}_1 = \begin{bmatrix} -3 & 4 \\ 4 & -1 \end{bmatrix} \begin{Bmatrix} -2 \\ 9 \end{Bmatrix}$$

$$= \begin{bmatrix} -3 \times -2 + 4 \times 9 \\ 4 \times -2 - 1 \times 9 \end{bmatrix}$$

$$= \begin{Bmatrix} 42 \\ -17 \end{Bmatrix}$$

$$\alpha_0 = \frac{\begin{Bmatrix} -2 \\ 9 \end{Bmatrix}^T \begin{Bmatrix} 42 \\ -17 \end{Bmatrix}}{\begin{Bmatrix} -2 \\ 9 \end{Bmatrix}^T \begin{Bmatrix} -2 \\ 9 \end{Bmatrix}}$$

$$= \frac{-2 \times -2 + 9 \times 9}{-2 \times -2 + 9 \times 9}$$

$$= -0.359$$

Step 5:

$$\{x\}_1 = \{x\}_0 + \alpha_0 \{v\}_1$$

$$= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + (-0.359) \begin{Bmatrix} -2 \\ 9 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.7173 \\ -3.229 \end{Bmatrix}$$

$$\{x\}_1 = \{x\}_0 - \alpha_0 \{z\}_0$$

$$= \begin{Bmatrix} -2 \\ 9 \end{Bmatrix} - (-0.359) \begin{Bmatrix} 42 \\ -17 \end{Bmatrix}$$

$$= \begin{Bmatrix} 13.078 \\ 2.897 \end{Bmatrix}$$

Step 6:

$\{x\}$ (we want it to converge to zero)

Step 7:

$$B_0 = \frac{\{x\}_0^T \{x\}_1}{\{x\}_0^T \{x\}_0}$$

$$= \frac{\begin{Bmatrix} 13.078 & 2.897 \end{Bmatrix} \begin{Bmatrix} -2 \\ 9 \end{Bmatrix}}{\begin{Bmatrix} 13.078 & 2.897 \end{Bmatrix} \begin{Bmatrix} 13.078 \\ 2.897 \end{Bmatrix}}$$

$$= \frac{-26.156 + 26.073}{13.078^2 + 2.897^2}$$

$$= 2.111$$

Step 8:

$$\{v\}_1 = \{x\}_1 + B_0 \{v\}_0$$

$$= \begin{Bmatrix} 13.078 \\ 2.897 \end{Bmatrix} + 2.111 \begin{Bmatrix} -2 \\ 9 \end{Bmatrix}$$

$$= \begin{Bmatrix} 8.856 \\ 21.896 \end{Bmatrix}$$

Next iteration,
 $i = 0 + 1 = 1$

Step 9:

$$\alpha_1 = \frac{\{v\}_1^T \{x\}_1}{\{v\}_1^T \{v\}_1}$$

$$\{z\}_1 = [A] \{v\}_1$$

$$= \begin{bmatrix} -3 & 4 \\ 4 & -1 \end{bmatrix} \begin{Bmatrix} 8.856 \\ 21.896 \end{Bmatrix}$$

$$= \begin{Bmatrix} 61.007 \\ 13.54 \end{Bmatrix}$$

$$\alpha_1 = \frac{\begin{Bmatrix} 13.078 & 2.897 \end{Bmatrix} \begin{Bmatrix} 61.007 \\ 13.54 \end{Bmatrix}}{\begin{Bmatrix} 8.856 & 21.896 \end{Bmatrix} \begin{Bmatrix} 61.007 \\ 13.54 \end{Bmatrix}}$$

$$= \frac{179.427}{836.567} = 0.21448$$

$$\{x\}_2 = \{x\}_1 + \alpha_1 \{v\}_1$$

$$= \begin{Bmatrix} 13.078 \\ 2.897 \end{Bmatrix} + 0.21448 \begin{Bmatrix} 8.856 \\ 21.896 \end{Bmatrix}$$

$$= \begin{Bmatrix} 2.6167 \\ 1.467 \end{Bmatrix}$$

$$\{a_3\}_2 = \{a_3\}_1 - a_1 \{a_2\}_1$$

$$= \begin{Bmatrix} 13.078 \\ 2.897 \end{Bmatrix} - 0.21448 \begin{Bmatrix} 61.007 \\ 13.54 \end{Bmatrix}$$

$$= \begin{Bmatrix} 13.078 \\ 2.897 \end{Bmatrix} - \begin{Bmatrix} 13.0817 \\ 2.9014 \end{Bmatrix}$$

$$= \begin{Bmatrix} -0.0037 \\ -0.004 \end{Bmatrix} \approx \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Step 11 $\therefore$ The solution is $\begin{Bmatrix} 2.6167 \\ 1.467 \end{Bmatrix}$

$$a_1 = 2.6167$$

$$a_2 = 1.467$$

Step 12 END

Q.

$$2a_1 - a_2 = 4$$

$$-a_1 + 2a_2 - a_3 = 0$$

$$-a_2 + 2a_3 = 0$$

$\rightarrow$ Solⁿ

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 0 \\ 0 \end{Bmatrix}$$

$$[A] \{a\} = \{b\}$$

$\therefore$ Matrix is symmetric.

conjugate gradient,

$$\begin{bmatrix} 5 & -2 & 0 \\ -2 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 10 \\ -10 \end{Bmatrix}$$

③

$f=0$

i) $\{x\}_0 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

← 8

(ii) $\{x\}_0 = \{b\} - [A]\{x\}_0 = \begin{Bmatrix} 20 \\ 10 \\ -10 \end{Bmatrix}$

(iii) $\{v\}_0 = \{x\}_0 = \begin{Bmatrix} 20 \\ 10 \\ -10 \end{Bmatrix}$

(iv) $\alpha_0 = \frac{\{x\}_0^T \{x\}_0}{\{v\}_0^T \{x\}_0} = \frac{600}{8000} = 0.3$

$$\{z\}_0 = [A]\{v\}_0 = \begin{Bmatrix} 80 \\ 0 \\ -40 \end{Bmatrix} \leftarrow \textcircled{2}$$

(v) $\{x\}_{i+1} = \{x\}_i + \alpha_i \{v\}_i$

$$= \begin{Bmatrix} 6 \\ 3 \\ -3 \end{Bmatrix} \leftarrow \textcircled{5}$$

(vi) $\{x\}_{i+1} = \{x\}_i - \alpha_i \{z\}_i = \begin{Bmatrix} -4 \\ 10 \\ 0 \end{Bmatrix} \leftarrow c$

(v) Not in permissible

(vii) $B_{i+1} = B_i = \frac{\{x\}_i^T \{x\}_i}{600} = \frac{120}{600} = 0.2$

(viii) $\{v\}_{i+1} = \{x\}_{i+1} - B_{i+1} \{x\}_i = \begin{Bmatrix} 0 \\ 10 \\ 0 \end{Bmatrix}$

Next: $j=1$

$$\textcircled{1} \quad \alpha_1 = \frac{\{x\}_1^T \{x\}_1}{\{v\}_1^T \{z\}_1} = \frac{120}{720} = \underline{0.16667}$$

$$\{z\}_1 = [A] \{v\}_1 = \begin{Bmatrix} -24 \\ 60 \\ 12 \end{Bmatrix} \quad \textcircled{D}$$

$$\textcircled{2} \quad \{z\}_2 = \{x\}_1 + \alpha_1 \{v\}_1$$

$$= \begin{Bmatrix} 6 \\ 3 \\ -1 \end{Bmatrix} + 0.16667 \begin{Bmatrix} 0 \\ 12 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 6 \\ 5 \\ -3 \end{Bmatrix}$$

$$\{x\}_2 = \{x\}_1 - \alpha_1 \{z\}_1$$

$$= \begin{Bmatrix} -4 \\ 10 \\ 2 \end{Bmatrix} - 0.16667 \begin{Bmatrix} -24 \\ 60 \\ 12 \end{Bmatrix}$$

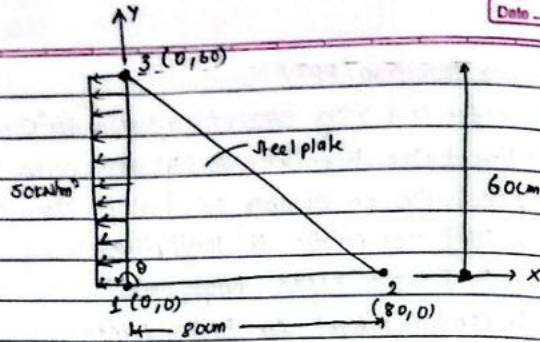
$$= \begin{Bmatrix} 8 \times 10^{-5} \\ -2 \times 10^{-4} \\ -4 \times 10^{-6} \end{Bmatrix}$$

residual is within permissible

Thus,

$$\begin{Bmatrix} x_1 = 5 \\ x_2 = 5 \\ x_3 = -3 \end{Bmatrix}$$

(9)



$$E = 210 \times 10^3 \text{ MPa}$$

$$G = 105 \times 10^3 \text{ MPa}$$

(Plane stress)

$$\gamma_{\text{steel}} = 78.5 \text{ kN/m}^3$$

$$t_{\text{steel}} = 10 \text{ mm}$$

Surface traction on side 1-3 $\Rightarrow q = -50 \text{ kN/m}$.

(1) Body forces,

$$\begin{Bmatrix} f_{1x}^b \\ f_{1y}^b \\ f_{1z}^b \end{Bmatrix} = \frac{\gamma_b A t}{3} \begin{Bmatrix} -1 \\ -1 \\ -1 \end{Bmatrix}$$

$$\begin{aligned} \gamma_b &= 78.5 \times \left(\frac{1}{2} \times 0.8 \times 0.6 \right) \times (0.01) \\ &= 0.188 \text{ kN} \\ &= 0.0628 \text{ kN} \end{aligned}$$

$$= 0.0628 \text{ kN}$$

(2) Traction force.

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{1z} \\ f_{1y} \\ f_{1z} \end{Bmatrix} = \frac{t \times l_{13}}{6} \begin{Bmatrix} 2P_1 \cos \theta + P_3 \cos \theta \\ 2P_1 \sin \theta + P_3 \sin \theta \\ P_1 \cos \theta + 2P_2 \cos \theta \\ P_1 \sin \theta + 2P_2 \sin \theta \end{Bmatrix}$$

$$\theta = 180^\circ$$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{1z} \\ f_{1y} \\ f_{1z} \end{Bmatrix} = 11 \times 10^{-3} \times 50 \begin{Bmatrix} -3 \\ 0 \\ -3 \\ 0 \end{Bmatrix} = -0.15 \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \text{ kN}$$

(3) Force vector

$$\{F\} = \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{1z} \\ f_{1y} \\ f_{1z} \end{Bmatrix} = \begin{Bmatrix} R_{1x} - 0.15 \\ R_{1y} - 0.0628 \\ R_{2x} \\ R_{2y} - 0.0628 \\ -0.15 \\ -0.0628 \end{Bmatrix} \text{ kN}$$

(4) Displacement matrix

$$\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \end{Bmatrix}$$

(5) Stiffness matrix:

$$[K] = A^T [B]^T [D] [B]$$

$$[B] = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

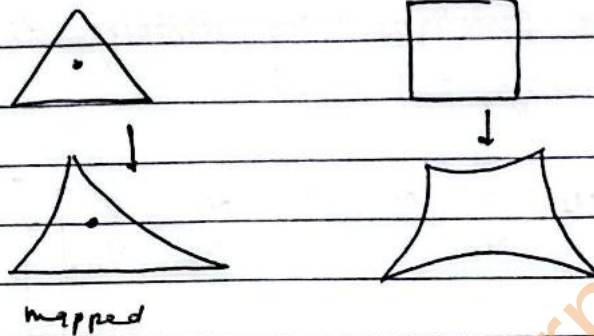
$$= \frac{1}{2A} \begin{bmatrix} -0.6 & 0 & +0.6 & 0 & 0 & 0 \\ 0 & -0.8 & 0 & 0 & 0 & 0.8 \\ -0.8 & -0.6 & 0 & 0.6 & 0.8 & 0 \end{bmatrix}$$

(a)

(b)

Q.3 contd

Isoparametric Function Formulation



* Isoparametric Formulation of 3-noded triangular element

(Assembling stiffness matrix using isoparametric formulation)

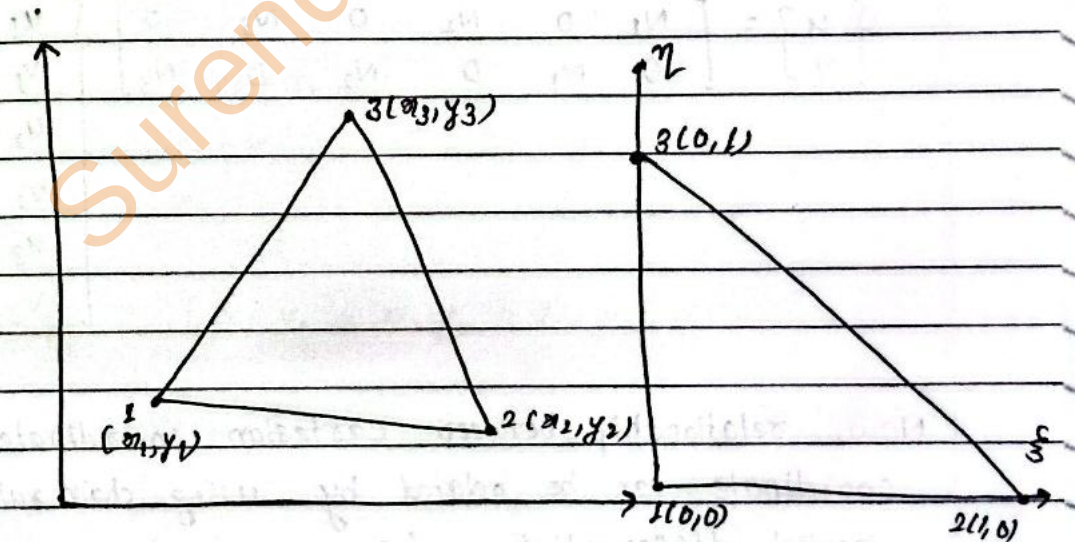


fig: Isoparametric element & typical parent element.

For parent element

shape functions are:

$$N_1 = 1 - \xi - \eta$$

$$N_2 = \xi$$

$$N_3 = \eta$$

①

We use shape functions for defining displacement as well as geometry i.e.

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{Bmatrix} \quad \text{--- (II)}$$

and,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad \text{--- (III)}$$

Now, relationship between cartesian co-ordinate & natural coordinate can be related by using chain rule of partial differentiation i.e.

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi} \cdot \frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi} \cdot \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta} \cdot \frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta} \cdot \frac{\partial}{\partial y}$$

$$\text{i.e. } \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$

$$= [J] \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \quad \text{--- (IV)}$$

here,

[J] is jacobian matrix.

[It relates derivative of the function in local coordinate system to derivative in global coordinate system]

where,

$$\frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi} [N_1 x_1 + N_2 x_2 + N_3 x_3]$$

$$= -x_1 + x_2$$

$$= x_2 - x_1$$

$$= x_{21}$$

$$\frac{\partial y}{\partial \xi} = \frac{\partial}{\partial \xi} [N_1 y_1 + N_2 y_2 + N_3 y_3]$$

$$= -y_1 + y_2$$

$$= y_2 - y_1$$

$$= y_{21}$$

$$\frac{\partial x}{\partial \eta} = \frac{\partial}{\partial \eta} [N_1 x_1 + N_2 x_2 + N_3 x_3]$$

$$= x_{31}$$

$$\begin{aligned} \frac{\partial u}{\partial \eta} &= \frac{\partial}{\partial \eta} [N_1 u_1 + N_2 u_2 + N_3 u_3] \\ &= u_2 - u_1 \\ &= u_{31} \end{aligned}$$

hence,

$$[J] = \begin{bmatrix} x_{21} & y_{21} \\ x_{31} & y_{31} \end{bmatrix} \quad \text{--- (V)}$$

Hence,

$$\begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} = \begin{bmatrix} x_{21} & y_{21} \\ x_{31} & y_{31} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

$$\text{or } \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} x_{21} & y_{21} \\ x_{31} & y_{31} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \frac{1}{x_{21} y_{31} - x_{31} y_{21}} \begin{bmatrix} y_{31} & -y_{21} \\ -x_{31} & x_{21} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{31} & -y_{21} \\ -x_{31} & x_{21} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} \quad \text{--- (VI)}$$

∴ We can express the relation for derivative of displacement as:

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{31} & -y_{21} & 0 & 0 \\ -x_{31} & x_{21} & 0 & 0 \\ 0 & 0 & y_{31} & -y_{21} \\ 0 & 0 & -x_{31} & x_{21} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} \quad \text{--- (VII)}$$

Strain-displacement relationship is given by

$$\{\epsilon\} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} \quad \text{--- (VIII)}$$

$$\therefore \{\epsilon\} = \frac{1}{2A} \begin{bmatrix} y_{31} & y_{12} & 0 & 0 \\ 0 & 0 & x_{13} & x_{21} \\ x_{13} & x_{21} & y_{31} & y_{12} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} \quad \text{--- (VIII)}$$

where,

$$\frac{\partial u}{\partial \xi} = \frac{\partial}{\partial \xi} [N_1 u_1 + N_2 u_2 + N_3 u_3] = -u_1 + u_2$$

$$\frac{\partial u}{\partial \eta} = -u_1 + u_2$$

$$\frac{\partial v}{\partial \xi} = -v_1 + v_2$$

$$\frac{\partial v}{\partial \eta} = -v_1 + v_2$$

$$\therefore \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} \quad \text{--- (IX)}$$

where,

$$\{B\} = \frac{1}{2A} \begin{bmatrix} y_{13} + y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{23} + x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} + x_{13} & y_{13} + y_{31} & x_{12} & y_{31} & x_{21} & y_{12} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} y_{13} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{23} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{12} & x_{12} & y_{31} & x_{21} & y_{12} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\therefore \{B\} = [B] \{u\} \quad \text{--- (3)}$$

FDM

↳ technique to solve DE

↳ domain represented by array of regularly spaced grid points

↳ DE converted to difference equation at a grid pt.

→ Derivative $\approx$ slope of line joining two points } Real derivative
Approx $\left[\frac{df}{dx} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right]$ } tangent at given pt.

→ FD approx. of first order derivative: (Taylor's series expansion)

(i) F/w difference

$$f(x + \Delta x) = f(x) + \Delta x \frac{\partial f}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 f}{\partial x^2} + \frac{\Delta x^3}{3!} \frac{\partial^3 f}{\partial x^3} + \dots \quad \text{--- (1)}$$

Truncation $\left\{ \text{Neglecting second and higher order terms} \right\}$

$$\left[\frac{\partial f}{\partial x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \right] \approx$$

(ii) B/w difference

$$f(x - \Delta x) = f(x) - \Delta x \frac{\partial f}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 f}{\partial x^2} - \frac{\Delta x^3}{3!} \frac{\partial^3 f}{\partial x^3} + \dots \quad \text{--- (11)}$$

Truncating

$$\left[\frac{\partial f}{\partial x} = \frac{f(x) - f(x - \Delta x)}{\Delta x} \right]$$

(iii) Central diff.

eqn (1) - eqn (11)

$$\left[\frac{\partial f}{\partial x} = \frac{f(x + \Delta x) - f(x - \Delta x)}{2 \Delta x} \right]$$

* Second order derivative

eqn (1) + eqn (11)

$$\left[\frac{\partial^2 f}{\partial x^2} = \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{\Delta x^2} \right]$$

Steps in FDM

- (i) Discretization of domain
- (ii) Discretization of PDE
- (iii) Specification of solution algorithm

* Order of accuracy of scheme

↳ represents how well the solution of finite difference method approximates the real solution of diff. eqn.

For Fw diff

$$\underbrace{\frac{f(x+\Delta x) - f(x)}{\Delta x}}_{\text{FDM}} - \underbrace{\frac{\partial f}{\partial x}}_{\text{Real}} = \frac{\Delta x}{2} \frac{\partial^2 f}{\partial x^2} + \frac{\Delta x^2}{6} \frac{\partial^3 f}{\partial x^3} + \dots$$

FTS = truncation error

Lowest power of $\Delta x \rightarrow$ determines order of accuracy

Here

1st order

As, $\Delta x \rightarrow 0$, error in higher order scheme tends to zero more rapidly than lower order scheme.

For Bw diff

↳ 1st order accurate.

central diff

$$f(x+\Delta x) - f(x-\Delta x) = 2\Delta x \frac{\partial f}{\partial x} + \frac{\Delta x^3}{3} \frac{\partial^3 f}{\partial x^3} + \dots$$

Divide by $2\Delta x$ & rearrange.

$$\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} - \frac{\partial f}{\partial x} = \frac{\Delta x^2}{6} \frac{\partial^3 f}{\partial x^3} + \dots$$

2nd order accurate

Consistency

↳ If continuous PDE (real) are obtained from FDE as Δx & $\Delta t \rightarrow 0$

eg: $\frac{\partial \phi}{\partial x} + \frac{\partial A}{\partial t} = 0$

say Fw diff

$$\frac{\phi(x+\Delta x) - \phi(x)}{\Delta x} + \frac{A(t+\Delta t) - A(t)}{\Delta t} = 0 \quad \text{FDE eqn}$$

Taylor's series expansion

$$\frac{\phi(x+\Delta x) - \phi(x)}{\Delta x} - \frac{\partial \phi}{\partial x} = \frac{\Delta x}{2} \frac{\partial^2 \phi}{\partial x^2} + \frac{\Delta x^2}{6} \frac{\partial^3 \phi}{\partial x^3} + \dots$$

As, $\Delta x \rightarrow 0$, $\frac{\phi(x+\Delta x) - \phi(x)}{\Delta x} = \frac{\partial \phi}{\partial x}$

FDE $\xrightarrow{\Delta x \rightarrow 0}$ Continuous PDE

Similarly for, $\Delta t \rightarrow 0$ $\frac{A(t+\Delta t) - A(t)}{\Delta t} = \frac{\partial A}{\partial t}$

consistent

Convergence

↳ If solution of FDM approaches the true solution as Δx and Δt approaches to zero.

Stability

↳ If round-off and truncation errors do not accumulate to cause the solution to diverge, then the solution is said to be stable.

↳ For stability analysis $\Rightarrow$ Fourier analysis performed.

* Explicit scheme

- ↳ solution at $t=n+1$, obtained from conditions at $t=n$
- ↳ equations solved sequentially
- ↳ simpler to implement
- ↳ can become unstable
- ↳ less efficient than implicit
- ↳ not suitable for simulation over long time period.

* Implicit scheme

- ↳ solution at $t=n+1$, obtained from conditions at $t=n$ & $t=n+1$
- ↳ " " simultaneously
- ↳ more complicated
- ↳ stable for large computation steps
- ↳ much faster than explicit
- ↳ can handle channel geometry varying significantly from one section to next

Explicit FDE (use time line n)

$$\frac{F/w}{\partial f} = \frac{f_{i+1}^n - f_i^n}{\Delta x} \quad \left| \quad \frac{\partial f}{\partial x} = \frac{f_i^n - f_{i-1}^n}{\Delta x} \right| \quad \text{central} \quad \frac{\partial f}{\partial x} = \frac{f_{i+1}^n - f_{i-1}^n}{2\Delta x}$$

Implicit FDE (use time line $n+1$)

$$\frac{F/w}{\partial f} = \frac{f_{i+1}^{n+1} - f_i^{n+1}}{\Delta x} \quad \left| \quad \frac{\partial f}{\partial x} = \frac{f_i^{n+1} - f_{i-1}^{n+1}}{\Delta x} \right| \quad \text{central} \quad \frac{\partial f}{\partial x} = \frac{f_{i+1}^{n+1} - f_{i-1}^{n+1}}{2\Delta x}$$

* Unsteady, non-uniform flow equation in open channel: St. Venant Equations

- Unsteady flow: Flood waves; storm water flow; surge
- Assumptions in St. Venant eqns

- ↳ channel $\rightarrow$ prismatic
- ↳ Flow $\rightarrow$ 1-D
- ↳ Flow $\rightarrow$ gradually varying along channel
- ↳ channel $\rightarrow$ approx. st. line.
- ↳ Bottom slope of channel $\rightarrow$ small
- ↳ Resistance coeff. for steady uniform turbulent flow $\rightarrow$ applicable Manning's eqn
- ↳ Fluid $\rightarrow$ incompressible & constant density

FULL EQUATIONS

1. Continuity: Inflow - outflow = rate of change of storage

$$\left[\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q \right]$$

2. Momentum: Net force = rate of change of momentum (2nd law)

$$\left[\frac{\partial Q}{\partial t} + \frac{\partial \left(\frac{\rho Q^2}{A} \right)}{\partial x} + gA \left(\frac{\partial x}{\partial x} + S_f - S_0 + S_e \right) + W_f b - \beta q V_z = 0 \right]$$

$W_f \rightarrow$ wind shear force

$S_e \rightarrow$ eddy loss slope

$\beta \rightarrow$ Momentum coefficient (or correction factor)

$V_z \rightarrow$ velocity of lateral inflow in the dirⁿ of channel flow

$q \rightarrow$ lateral flow per unit length along the channel

Simplified & most widely used (conservative forms)

$\rightarrow$ take $\beta = 1$, neglect: eddy loss, wind shear & lateral flow

$$1. \quad \left[\frac{\partial Q}{\partial t} + \frac{\partial A}{\partial t} = 0 \right] \quad \text{--- ①}$$

$$2. \quad \left[\frac{\partial Q}{\partial t} + \frac{\partial \left(\frac{Q^2}{A} \right)}{\partial x} + gA \frac{\partial x}{\partial x} - gA (S_0 - S_f) = 0 \right] \quad \text{--- ②}$$

- 1st term: Inertial force due to local accelⁿ
- 2nd term: " " " " convective "
- 3rd term: pressure "
- 4th term: Gravity "
- 5th term: Friction "

Unknowns: Q & $y \rightarrow$ conservative form
 V & $y \rightarrow$ non-conservative form

Independent variables: time (t)
distance along the channel (x)

Simplifications of St. Venant equations

1. Dynamic Wave Model.

- ↳ Full Momentum eqⁿ (i.e. eqⁿ (II)) used
- ↳ (I) & (II)
- ↳ Flood wave in wide river
- ↳ Rivers where flood modified by development of stage

2. Diffusive wave model

- ↳ Neglect 1st & 2nd term of eqⁿ (II)
- ↳ Accelⁿ term small than water surface slope for slow rising flood wave.

$$\left[\frac{\partial y}{\partial x} = S_0 - S_f \right] \rightarrow \text{(II)*}$$

- ↳ represent backwater effect
- ↳ steep slope channel with flow influenced by bed roughness

3. Kinematic wave Model.

- ↳ Neglect 1st & 3rd term.

$$S_0 = S_f \rightarrow \text{(II)*} \quad (\text{gravity \& friction force balance})$$

continuity eqⁿ describes

Momentum eqⁿ in KWM

$$A = \alpha Q^\beta \rightarrow \text{(III)}$$

$$\frac{\partial A}{\partial t} = \alpha \beta Q^{\beta-1} \frac{\partial Q}{\partial t} \rightarrow \text{(IV)}$$

Then

continuity eqⁿ

$$\left[\frac{\partial Q}{\partial x} + \alpha \beta Q^{\beta-1} \frac{\partial Q}{\partial t} = q \right] \rightarrow \text{(I)***}$$

Manning's eqⁿ

$$Q = \frac{1}{n} \frac{A^{5/3}}{P^{2/3}} S^{1/2}$$

$$\Rightarrow A = \left(\frac{n P^{2/3}}{\sqrt{S}} \right)^{3/5} Q^{3/5}$$

$\uparrow$
 α

$A = \alpha Q^\beta$

St. Venant Non-conservative form

$$\text{Continuity: } \frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

$$\text{or } \frac{\partial (AV)}{\partial x} + \left(\frac{\partial A}{\partial y} \right) \frac{\partial y}{\partial t} = 0$$

$$\text{or } A \frac{\partial V}{\partial x} + V \frac{\partial A}{\partial x} + T \frac{\partial y}{\partial t} = 0$$

$$\text{or } A \frac{\partial V}{\partial x} + VT \frac{\partial y}{\partial x} + T \frac{\partial y}{\partial t} = 0$$

Divide by T

$$\left[D \frac{\partial V}{\partial x} + V \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = 0 \right]$$

$$D = \frac{A}{T} = \text{hydraulic depth}$$

Momentum

$$\frac{\partial(AV)}{\partial t} + \frac{\partial(QV)}{\partial x} + gA \frac{\partial x}{\partial x} - gA(S_0 - S_f) = 0$$

$$\left(\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \right) \text{ \& divide throughout by } A$$

$$\frac{\partial}{\partial t} \left[\frac{\partial V}{\partial t} + \frac{V \partial V}{\partial x} + g \frac{\partial x}{\partial x} - g(S_0 - S_f) \right] = 0$$

Numerical scheme (FDM) for KWM

↳ Channel properties; initial condition & Boundary condition should be known

↳ Formulated in

• Linear: kinematic eqs in terms of Q used

• Non-linear: original continuity & alternative form of momentum eqs used

A. Linear scheme (KWM)

1. First order accurate
F/W or B/W

implicit scheme

For $t=n+1$ use $t=n$

Non-derivative term:

$$Q = \frac{Q_{i+1}^n + Q_i^{n+1}}{2}; \text{ (avg. along } n \text{ diagonal)}$$

$$q = \frac{q_{i+1}^{n+1} + q_i^n}{2}; \text{ (avg. along } i+1 \text{ distance line)}$$

$$\left[Q_{i+1}^{n+1} = \frac{\frac{\Delta t}{\Delta x} Q_i^{n+1} + \alpha \beta Q_{i+1}^n \left(\frac{Q_{i+1}^n + Q_{i+1}^{n+1}}{2} \right)^{\beta-1} + \left(\frac{q_{i+1}^{n+1} + q_i^n}{2} \right) \Delta t}{\left(\frac{\Delta t}{\Delta x} + \alpha \beta \left(\frac{Q_{i+1}^n + Q_{i+1}^{n+1}}{2} \right)^{\beta-1} \right)} \right] \text{--- (a)}$$

For explicit: discretize at n time interval. level.

2. Second order accurate scheme (central diff: explicit)
central diff.
For $t=n+1$ use $t=n$ condition.

Non-derivative term:

$$Q = \frac{Q_{i+1}^{n+1} + Q_i^n + Q_{i-1}^{n+1}}{3} \text{ (avg. along known diagonal)}$$

$$q = \frac{q_i^{n+1} + q_i^n + q_i^{n-1}}{3} \text{ (avg. along unknown distance line)}$$

Solve for Q_i^{n+1}

B. Non-linear (KWM)

$$\left[\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q \right] \text{--- (I)} \text{ \& } [A = \alpha Q^\beta] \text{--- (II)}$$

1. First order accurate implicit

$$A_{i+1}^{n+1} = \alpha (Q_{i+1}^{n+1})^\beta \text{ and } A_{i+1}^n = \alpha (Q_{i+1}^n)^\beta$$

Eq is non-linear in Q_{i+1}^{n+1}

(NR).

$$\text{Residual error} = f(Q_{i+1}^{n+1})$$

$$f(Q_{i+1}^{n+1}) = \frac{\Delta t}{\Delta x} Q_{i+1}^{n+1} + \alpha (Q_{i+1}^{n+1})^\beta - \left(\frac{\Delta t}{\Delta x} Q_i^{n+1} + \alpha (Q_{i+1}^n)^\beta + \Delta t \left(\frac{q_{i+1}^{n+1} + q_i^n}{2} \right) \right) \text{--- (IV)}$$

known

$$f'(Q_{i+1}^{n+1}) = \frac{\Delta t}{\Delta x} + \alpha \beta (Q_{i+1}^{n+1})^{\beta-1} \text{--- (V)}$$

$$(Q_{i+1}^{n+1})_{k+1} = (Q_{i+1}^{n+1})_k - \frac{f(Q_{i+1}^{n+1})_k}{f'(Q_{i+1}^{n+1})_k} \text{--- (VI)}$$

Initial value
↳ arbitrary
or
↳ eq = 0
or
↳ avg. along known dia.

Non-linear scheme for kinematic wave model

① First order accurate implicit scheme

Eq to solve:

$$\frac{\partial \phi}{\partial x} + \frac{\partial A}{\partial t} = q$$

To find: ϕ_{i+1}^{n+1}

~ (PD → derivative term को discrete करें)
Derivative term (discretization):

$$\sim \left[\frac{\partial \phi}{\partial x} = \frac{\phi_{i+1}^{n+1} - \phi_i^{n+1}}{\Delta x} \right] \text{ (backward)}$$

$$\sim \left[\frac{\partial A}{\partial t} = \frac{A_{i+1}^{n+1} - A_{i+1}^n}{\Delta t} \right]$$

Non-derivative term:

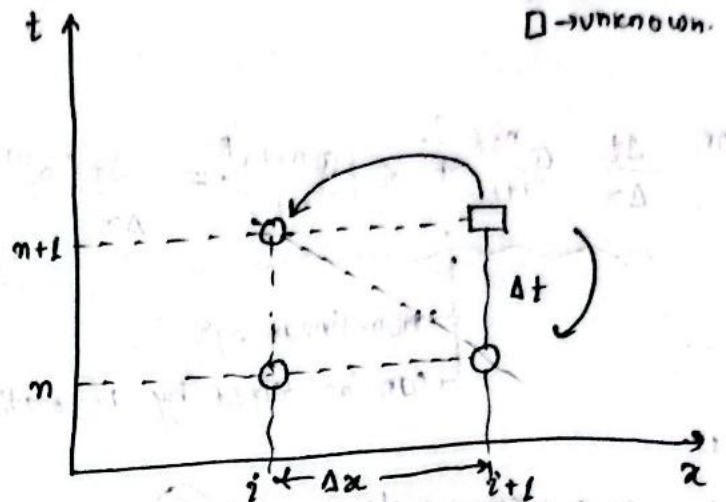
$$\sim \left[q = \frac{q_i^{n+1} + q_{i+1}^n}{2} \right]$$

Substituting.

$$\frac{\phi_{i+1}^{n+1} - \phi_i^{n+1}}{\Delta x} + \frac{A_{i+1}^{n+1} - A_{i+1}^n}{\Delta t} = \frac{q_i^{n+1} + q_{i+1}^n}{2}$$

We know

$$A = \alpha \phi^\beta$$



Implicit: पूरा line मा unknown है उसे solve करें
n+1

~ explicit: n (तबको line) solve करें

~ (time को respect मा der. है अतः unknown को solve करने (no need to see implicit or explicit))

(known तरफ को diagonal को Avg. लिने)

(A लाई पनि ϕ को term मा लाने)

$$\frac{Q_{i+1}^{n+1} - Q_i^{n+1}}{\Delta x} + \frac{\alpha (Q_{i+1}^{n+1})^\beta - \alpha (Q_i^n)^\beta}{\Delta t} = \frac{Q_i^{n+1} + Q_{i+1}^{n+1}}{2} \quad \left(\begin{array}{l} \text{unknown } Q_{i+1}^{n+1} \\ \text{power diff} \\ \Delta t \end{array} \right)$$

(Non-linear eq)
NR method

$$\text{or, } \frac{\Delta t}{\Delta x} Q_{i+1}^{n+1} + \alpha (Q_{i+1}^{n+1})^\beta = \frac{\Delta t}{\Delta x} Q_i^{n+1} + \alpha (Q_i^n)^\beta + \Delta t \left(\frac{Q_i^{n+1} + Q_{i+1}^{n+1}}{2} \right)$$

Non-linear eq.

can be solved by iteration using Newton Raphson Method.

RHS → Known term → (C)

LHS → Unknown

$$\text{Residual error: } f(Q_{i+1}^{n+1}) = \frac{\Delta t}{\Delta x} Q_{i+1}^{n+1} + \alpha (Q_{i+1}^{n+1})^\beta - C \quad \text{--- (i)}$$

where C = RHS part

$$f'(Q_{i+1}^{n+1}) = \frac{\Delta t}{\Delta x} + \alpha \beta (Q_{i+1}^{n+1})^{\beta-1} \quad \text{--- (ii)}$$

Newton-Raphson formula:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$(Q_{i+1}^{n+1})_{k+1} = (Q_{i+1}^{n+1})_k - \frac{f(Q_{i+1}^{n+1})_k}{f'(Q_{i+1}^{n+1})_k} \quad \text{--- (iii)}$$

Any arbitrary value can be taken as initial value, However, for faster convergence, it is better to use Q_{i+1}^{n+1} obtained from linear scheme as initial condition.

Q. Rectangular channel,
width = 30m (b)
S = 0.01 (bed slope)
n = 0.035

$$Q_i^{n+1} = 30 \text{ m}^3/\text{s}, Q_i^n = 22 \text{ m}^3/\text{s}, Q_{i+1}^n = 20 \text{ m}^3/\text{s}$$

$$\Delta x = 1500 \text{ m and } \Delta t = 10 \text{ min.} = 600 \text{ sec.}$$

$$Q_{i+1}^{n+1} = ?$$

→ Soln:

$$q = \text{lateral inflow} = 0 \text{ (as not given)}$$

From linear KW Model,

$$\alpha = 1.838$$

$$\beta =$$

For non-linear kinematic wave model, (implicit)
Use NR

$$\text{Residual error} = f(Q_{i+1}^{n+1}) = \frac{600}{1500} Q_{i+1}^{n+1} + 1.838 (Q_{i+1}^{n+1})^{0.6} - 23.09 \quad \text{--- (c)}$$

$$f'(Q_{i+1}^{n+1}) = \frac{600}{1500} + 1.838 \times 0.6 (Q_{i+1}^{n+1})^{-0.4}$$

From NR method, (solve by iteration)

$$(Q_{i+1}^{n+1})_{k+1} = (Q_{i+1}^{n+1})_k - \frac{f(Q_{i+1}^{n+1})_k}{f'(Q_{i+1}^{n+1})_k}$$

say, taking trial Q known at t diagonal Q at $t + \Delta t$
avg $\frac{20+30}{2} = 25$

Better if take Q by linear scheme

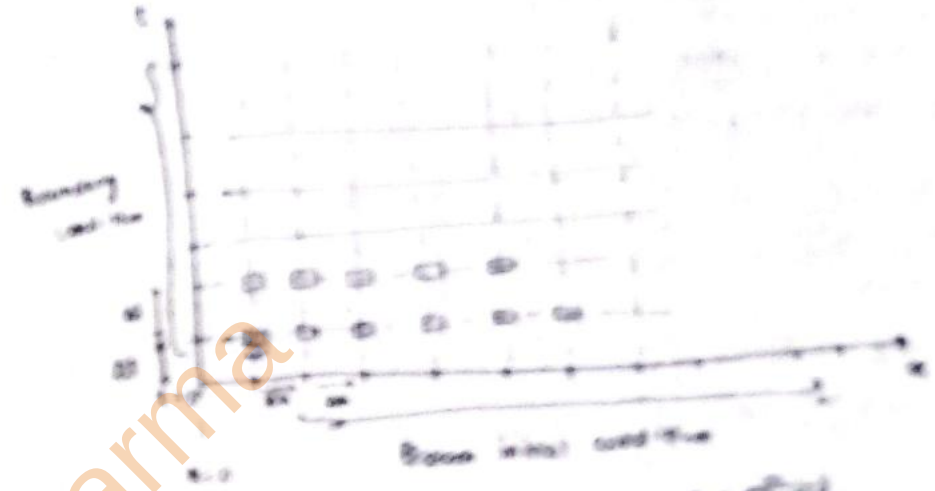
Iteration table:

Trial no	Total value $(Q_{i+1}^{n+1})_k$	$f(Q_{i+1}^{n+1})_k$	$f'(Q_{i+1}^{n+1})_k$	$(Q_{i+1}^{n+1})_{k+1}$
1	25	0.000000 -0.403	0.7043	25.5825
2	25.5825	-0.000769	0.70152	25.5836
3	25.5836	-0.0000695	0.70152	<u>25.5836</u>

Let $\left[\because \theta_{100}^{(1)} = 42.586^\circ \right] \rightarrow$ max energy

(for given phase, $\theta_{100}^{(1)} = 42.586^\circ$)

But while writing computer program (for above graph plotting)



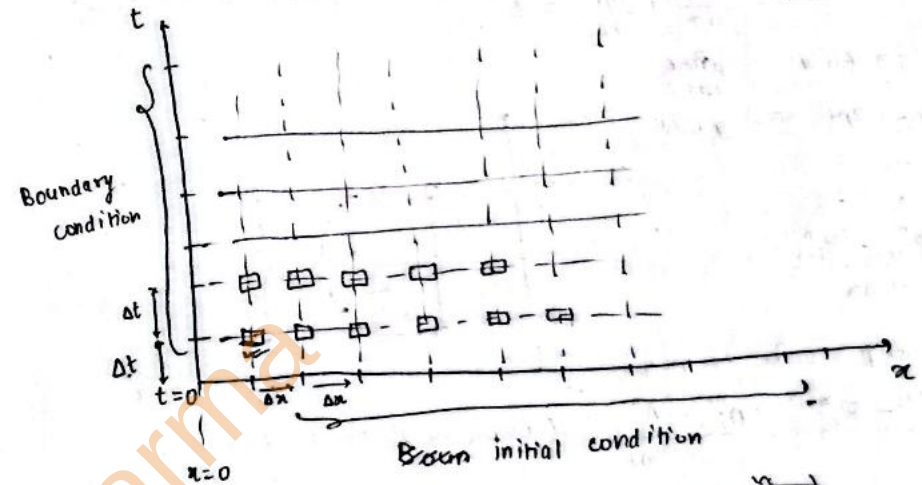
Scatter angle (deg) vs. Scatter angle (deg)

$\left\{ \begin{array}{l} \theta_1 (0-90^\circ) \rightarrow \text{Scatter} \\ \theta_2 \rightarrow \text{Phase} \end{array} \right.$

$$\left[\because \phi_{i+1}^{n+1} = 25.5836 \text{ m/s} \right] \rightarrow \text{more reliable}$$

$$(\text{Ans from linear, } \phi_{i+1}^{n+1} = 25.78 \text{ m/s})$$

But while writing computer program (to solve complex problem)



unknown find ϕ_i^n ϕ_i^{n+1} unknown find ϕ_i^{n+1}

$$\begin{cases} I_c(t=0) \rightarrow \text{Known} \\ B_c \rightarrow \text{Known} \end{cases}$$

Algorithm

2nd-order accurate scheme for non-linear kinematic wave
(central difference scheme: explicit) $\hookrightarrow$ Leap-frog scheme
(अप्रत्यक्ष अंतरांतरण)

Eq to solve: $\frac{\partial \phi}{\partial x} + \frac{\partial A}{\partial t} = 0$

To find: ϕ_{i+1}^{n+1}
(2nd-order $\rightarrow$ 9pt. ϕ grid)

Discretization:

$$\frac{\partial \phi}{\partial x} = \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2 \Delta x}$$

(explicit: ϕ step $\rightarrow$ n -line)

$$\frac{\partial A}{\partial t} = \frac{\phi_{i+1}^{n+1} - \phi_{i-1}^{n+1}}{2 \Delta t} \quad (A_i^{n+1} - A_i^{n-1}) \quad (\text{unknown } A \text{ line } n+1 \text{ req.})$$

For non-derivative term, (known diagonal) [unknown ϕ at n to avg.]

$$q = \frac{q_{i+1}^{n+1} + q_i^{n+1} + q_{i-1}^{n+1}}{3}$$

Now substituting

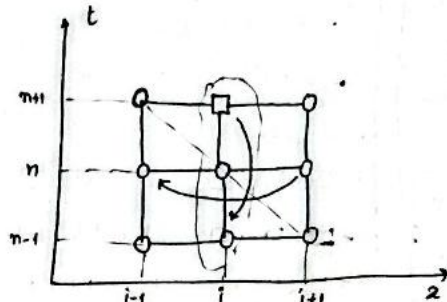
$$\frac{\phi_{i+1}^n - \phi_{i-1}^n}{2 \Delta x} + \frac{A_i^{n+1} - A_i^{n-1}}{2 \Delta t} = \frac{q_{i+1}^{n+1} + q_i^{n+1} + q_{i-1}^{n+1}}{3}$$

We know, $A = \alpha \phi^\beta$

$$\frac{\phi_{i+1}^n - \phi_{i-1}^n}{2 \Delta x} + \frac{\alpha (\phi_i^{n+1})^\beta - \alpha (\phi_i^{n-1})^\beta}{2 \Delta t} = \frac{q_{i+1}^{n+1} + q_i^{n+1} + q_{i-1}^{n+1}}{3}$$

$$\hookrightarrow (\phi_i^{n+1}) = (\text{known})^{1/\beta}$$

(no need to apply NR) $\rightarrow$ for explicit scheme



Dynamic Wave Model

(St. Venant's eqs)

Continuity eq: $\frac{\partial \phi}{\partial x} + \frac{\partial A}{\partial t} = 0 \quad \text{--- (I)}$

Momentum eq:

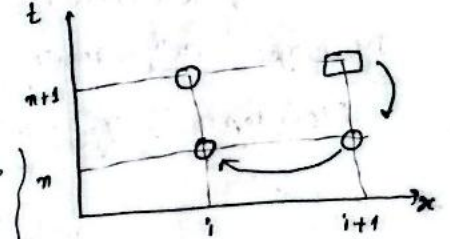
$$\frac{\partial \phi}{\partial t} + \frac{\partial (\phi^2/A)}{\partial x} + g A \frac{\partial y}{\partial x} - g A (S_0 - S_f) = 0 \quad \text{--- (II)}$$

1st order accurate explicit scheme:
(2x2 grid) (अप्रत्यक्ष 2x2 ग्रिड)

Eq to solve: (I) & (II)

Discretization:

$$\left\{ \begin{aligned} \frac{\partial f}{\partial x} &= \frac{f_{i+1}^n - f_i^n}{\Delta x} \\ \frac{\partial f}{\partial t} &= \frac{f_{i+1}^{n+1} - f_i^{n+1}}{\Delta t} \end{aligned} \right\} \quad \left\{ \begin{aligned} f &= f_{i+1}^n \text{ or } f_i^n \\ &= \frac{f_i^n + f_{i+1}^n}{2} \end{aligned} \right\} \quad \text{Non-derivative km}$$



$x \rightarrow$ step $\rightarrow$ explicit
 $t \rightarrow$ unknown ϕ line req.

Continuity eq:

$$\frac{\phi_{i+1}^n - \phi_i^n}{\Delta x} + \frac{A_{i+1}^{n+1} - A_i^{n+1}}{\Delta t} = 0$$

$$\Rightarrow A_{i+1}^{n+1} = A_i^{n+1} - \frac{\Delta t}{\Delta x} (\phi_{i+1}^n - \phi_i^n)$$

Momentum eq:

$$\frac{\phi_{i+1}^{n+1} - \phi_i^n}{\Delta t} + \frac{(\frac{\phi^2}{A})_{i+1}^n - (\frac{\phi^2}{A})_i^n}{\Delta x} + g \frac{A_{i+1}^n}{\Delta x} \frac{y_{i+1}^n - y_i^n}{\Delta x} - g A_{i+1}^n (S_0 - S_f)_{i+1}^n = 0$$

$$\frac{(\frac{\phi^2}{A})_{i+1}^n - (\frac{\phi^2}{A})_i^n}{\Delta x} + g \frac{A_{i+1}^n}{\Delta x} \frac{y_{i+1}^n - y_i^n}{\Delta x} - g A_{i+1}^n (S_0 - S_f)_{i+1}^n = 0$$

known ϕ at n

$$u_i^{n+1} = \frac{Q_i^{n+1}}{A_i^{n+1}}$$

Q. Using any explicit FD scheme for full Saint Venant equations compute discharge and flow depth at grid $(i, n+1)$ for the following data:

Rectangular channel, width = 10m, Bed slope = 0.0002,

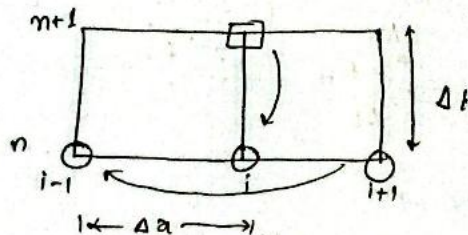
$n = 0.04$, NO lateral flow, $\Delta x = 1\text{km}$, $\Delta t = 5\text{min}$.

Discharge: $Q_{i-1}^n = 40\text{m}^3/\text{s}$, $Q_i^n = 38\text{m}^3/\text{s}$, $Q_{i+1}^n = 37.5\text{m}^3/\text{s}$.

Flow depth: $y_{i-1}^n = 1.9$, $y_i^n = 1.85$, $y_{i+1}^n = 2\text{m}$.

dynamic wave model

2nd order
3 given



→ Soln.

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \quad \text{--- (1) (continuity eq.)}$$

$$\text{or, } \frac{Q_{i+1}^n - Q_{i-1}^n}{2\Delta x} + \frac{A_i^{n+1} - A_i^n}{\Delta t} = 0$$

$$\text{or, } A_i^{n+1} = \frac{Q_{i+1}^n - Q_{i-1}^n}{2\Delta x} \Delta t + A_i^n$$

$$\therefore \frac{37.5 - 40}{2 \times 1000} + \frac{A_i^{n+1} - 1.85 \times 10}{5 \times 60} = 0$$

$$\Rightarrow [A_i^{n+1} = 18.875\text{m}^2]$$

$$y_i^{n+1} = \frac{18.875}{10} = 1.8875\text{m}$$

Momentum eq.:

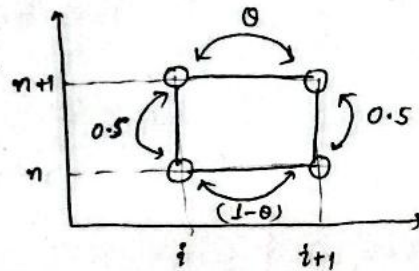
$$L \cdot Q_i^{n+1} =$$

Implicit scheme (Dynamic wave Model)

- ↳ Implicit four point scheme (for more accuracy/higher stability)
- ↳ Preissman scheme
- ↳ Box scheme.

we four point used to find $\frac{\partial f}{\partial x}$.

(By multiplying with weightage)
(explicit + implicit $\frac{\partial f}{\partial x}$ in
weightage at multiply
of in $\frac{\partial f}{\partial x}$)
(total weightage = 1)



Derivative term

$$\left\{ \begin{aligned} \frac{\partial f}{\partial x} &= \frac{f_{i+1}^{n+1} - f_i^{n+1}}{\Delta x} \times \theta + \frac{f_{i+1}^n - f_i^n}{\Delta x} \times (1-\theta) \\ \frac{\partial f}{\partial t} &= \frac{f_{i+1}^{n+1} - f_{i+1}^n}{\Delta t} \times 0.5 + \frac{f_i^{n+1} - f_i^n}{\Delta t} \times 0.5 \end{aligned} \right.$$

Non-derivative term (say, f)

$$f = \frac{f_{i+1}^{n+1} + f_i^{n+1}}{2} \times \theta + \frac{f_{i+1}^n + f_i^n}{2} \times (1-\theta)$$

- If, $\theta = 0.5$ $0.5 \times 0.5 \Rightarrow$ Box scheme
- If, $\theta = 0$; Fully explicit scheme
- If, $\theta = 1$; Fully implicit scheme
- Generally, $\theta = 0.5$ to 1 (unknown f_{i+1}^{n+1} weightage at t)

Continuity eqⁿ:

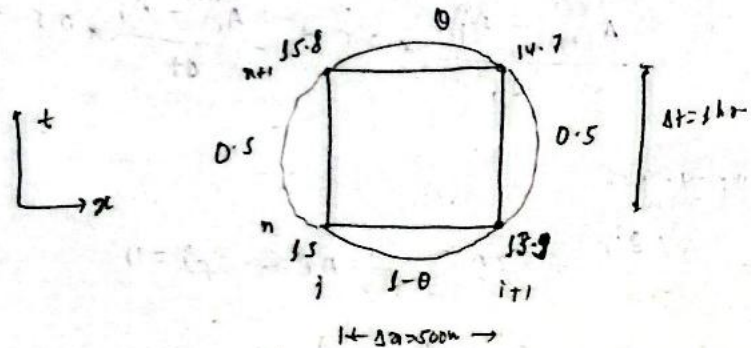
$$\frac{\partial \theta}{\partial x} + \frac{\partial A}{\partial t} = 0$$

$$\text{On } \frac{\theta_{i+1}^{n+1} - \theta_i^{n+1}}{\Delta x} \times \theta + \frac{\theta_{i+1}^n - \theta_i^n}{\Delta x} \times (1-\theta) + \frac{A_{i+1}^{n+1} - A_{i+1}^n}{\Delta t} \times 0.5 + \frac{A_i^{n+1} - A_i^n}{\Delta t} \times 0.5 = 0$$

Momentum eqⁿ:

$$\begin{aligned} \frac{\partial \theta}{\partial t} + \frac{\partial (\theta^2/A)}{\partial x} + gA \frac{\partial y}{\partial x} - gA(S_0 - S_f) &= 0 \\ \text{On } \left(\frac{\theta_{i+1}^{n+1} - \theta_{i+1}^n}{\Delta t} \times 0.5 + \frac{\theta_i^{n+1} - \theta_i^n}{\Delta t} \times 0.5 \right) + \\ &\left[\frac{(\frac{\theta^2}{A})_{i+1}^{n+1} - (\frac{\theta^2}{A})_i^{n+1}}{\Delta x} \times \theta + \frac{(\frac{\theta^2}{A})_{i+1}^n - (\frac{\theta^2}{A})_i^n}{\Delta x} \times (1-\theta) \right] \\ &+ g \left(\frac{A_{i+1}^{n+1} + A_i^{n+1}}{2} \times \theta + \frac{A_{i+1}^n + A_i^n}{2} \times (1-\theta) \right) \times \\ &\left(\frac{y_{i+1}^{n+1} - y_i^{n+1}}{\Delta x} \times \theta + \frac{y_{i+1}^n - y_i^n}{\Delta x} \times (1-\theta) \right) \\ &- g \left(\frac{A_{i+1}^{n+1} + A_i^{n+1}}{2} \times \theta + \frac{A_{i+1}^n + A_i^n}{2} \times (1-\theta) \right) \times \left(\frac{(S_0 - S_f)_{i+1}^{n+1} + (S_0 - S_f)_i^{n+1}}{2} \times \theta \right. \\ &\quad \left. + \frac{(S_0 - S_f)_{i+1}^n + (S_0 - S_f)_i^n}{2} \times (1-\theta) \right) = 0 \end{aligned}$$

Q. The value of flow rate at four points in the space-time grid are given below:
 $Q_i^n = 15 \text{ m}^3/\text{s}$, $Q_i^{n+1} = 15.8 \text{ m}^3/\text{s}$, $Q_{i+1}^n = 13.9 \text{ m}^3/\text{s}$ and $Q_{i+1}^{n+1} = 14.7 \text{ m}^3/\text{s}$.
 Taking $\Delta x = 500 \text{ m}$, $\Delta t = 1 \text{ hr}$ and $\theta = 0.55$, calculate the values of $\frac{\partial Q}{\partial x}$ and $\frac{\partial Q}{\partial t}$ using implicit four-point method.



Now,

$$\frac{\partial Q}{\partial x} = \frac{Q_{i+1}^{n+1} - Q_i^{n+1}}{\Delta x} * \theta + \frac{Q_{i+1}^n - Q_i^n}{\Delta x} * (1 - \theta)$$

$$= \frac{(14.7 - 15.8)}{500} * 0.55 + \frac{(13.9 - 15)}{500} * (1 - 0.55)$$

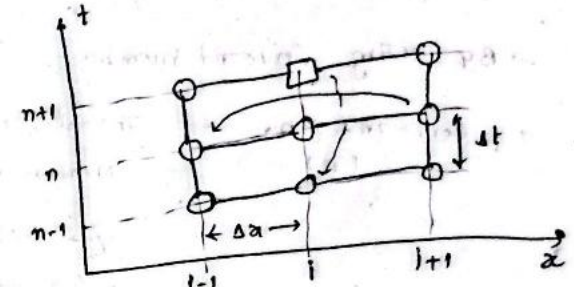
$$= -2.2 * 10^{-3} \text{ m}^3/\text{s}/\text{m}$$

$$\frac{\partial Q}{\partial t} = \frac{Q_{i+1}^{n+1} - Q_{i+1}^n}{\Delta t} * \theta + \frac{Q_i^{n+1} - Q_i^n}{\Delta t} * (1 - \theta)$$

$$= \frac{(14.7 - 13.9)}{1 \times 3600} * 0.55 + \frac{(15.8 - 15)}{1 \times 3600} * (1 - 0.55)$$

$$= 2.22 * 10^{-4} \text{ m}^3/\text{s}/\text{s}$$

Second-order accurate scheme (dynamic wave Model)
 ↳ Leap-frog scheme
 ↳ central
 (explicit scheme)



Derivative term

$$\frac{\partial f}{\partial x} = \frac{f_{i+1}^n - f_{i-1}^n}{2 \Delta x}; \quad \frac{\partial f}{\partial t} = \frac{f_i^{n+1} - f_i^{n-1}}{2 \Delta t}$$

Non-derivative term

(n-line) $f = f_i^n$ 'or' $\frac{f_{i-1}^n + f_{i+1}^n}{2}$ 'or' $\frac{f_{i-1}^n + f_i^n + f_{i+1}^n}{3}$

continuity $\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$

$$\frac{Q_{i+1}^n - Q_{i-1}^n}{2 \Delta x} + \frac{A_i^{n+1} - A_i^{n-1}}{2 \Delta t} = 0$$

$$\Rightarrow A_i^{n+1} = \dots$$

Momentum eqⁿ

$$\frac{\partial Q}{\partial t} + \frac{\partial (Q^2/A)}{\partial x} + gA \frac{\partial x}{\partial x} - gA (h_b - h_f) = 0$$

$$\text{or } \frac{Q_{i+1}^{n+1} + Q_{i-1}^{n+1}}{2 \Delta t} + \frac{(Q^2/A)_{i+1}^n - (Q^2/A)_{i-1}^n}{2 \Delta x} + gA \left(\frac{y_{i+1}^n - y_{i-1}^n}{2 \Delta x} \right) - gA_i^n (h_{b,i} - h_{f,i})^n = 0$$

$$Q_i^{n+1} = \dots$$

Stability of numerical solution

↳ explicit scheme stable
implicit is unstable to $\frac{\Delta x}{\Delta t}$

→ By using Courant number

$$\rightarrow \text{Courant no.} = \frac{\text{Theoretical velocity}}{\text{Numerical or actual vel.}}$$

$$C = \frac{V + |c|}{\Delta x / \Delta t}$$

→ For stability
Courant no. ≤ 1

$$\rightarrow \frac{1}{C} = \frac{\Delta x}{\Delta t} \leq V + |c|$$

$$\frac{\Delta x}{\Delta t} = c$$

$$\left[\Delta t = \frac{\Delta x}{c} \right]$$

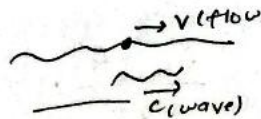
→ Δt acc. to taken Δx

→ celerity of dynamic wave (c) = $\sqrt{gy}$ (rec. channel)

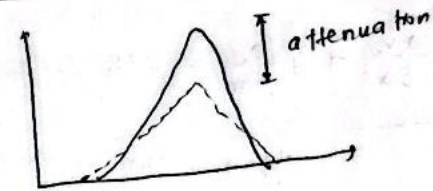
→ For other channel = $c = \sqrt{g \cdot A/T}$

A → x-section area
T → top-width

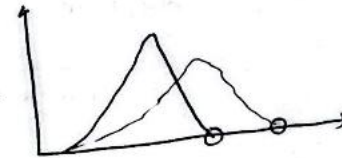
$$\rightarrow c = \sqrt{\frac{E/S}{1 + \frac{Dk}{EF}}} \quad (\text{pipe flow})$$



Diffusion
(Amplitude error)
(due to truncation error)



Dispersion
(timing of peak error)



→ St. Venant eq. (Non-conservative form)

↳ in terms of v & y

$$Q = A \cdot v = b \cdot y \cdot v$$



Continuity eq.

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

$$\text{or } \frac{\partial (byv)}{\partial x} + \frac{\partial (by)}{\partial t} = 0$$

$$\text{or } \left[v \frac{\partial y}{\partial x} + y \frac{\partial v}{\partial x} + \frac{\partial y}{\partial t} = 0 \right]$$

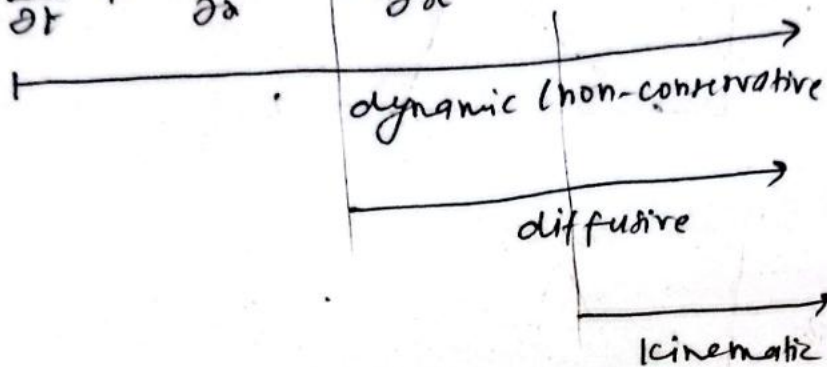
Momentum eq.

$$\frac{\partial Q}{\partial t} + \frac{\partial (Q^2/A)}{\partial x} + gA \frac{\partial y}{\partial x} - gA (s_0 - s_f) = 0$$

$$\text{or } \frac{\partial (byv)}{\partial t} + \frac{\partial}{\partial x}$$

$$\text{or } \left[\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + g \frac{\partial y}{\partial x} - g (s_0 - s_f) = 0 \right]$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + g \frac{\partial \eta}{\partial x} - g(\delta_0 - \delta_f) = 0$$



Surendra Sharma

Q.1

Channel

$$b = 40 \text{ m}$$

$$S_0 = 2\% = 0.02$$

$$n = 0.03$$

$$Q = 100 \text{ m}^3/\text{s}$$

$$\Delta x = 1500 \text{ m}$$

Recommend the maximum time step for stable condition of kinematic wave routing in this condition. Assume hydraulic radius is equal to flow depth.

$$\Delta t = 2$$

→ Soln:

$$R = y$$

For stability

Courant number ≤ 1

$$\frac{v + |c|}{\frac{\Delta x}{\Delta t}} \leq 1$$

For $v \ll c$

$$\frac{c_k}{\frac{\Delta x}{\Delta t}} \leq 1$$

$[c_k \rightarrow \text{kinematic wave celerity}]$

Using Manning's eq:

$$Q = \frac{1}{n} A R^{2/3} S^{1/2}$$

$$\text{or } Q = \frac{1}{0.03} (by) y^{2/3} \times 0.02^{1/2}$$

40m

$$[Q = 188.56 y^{5/3}] \text{ --- (1)}$$

For $Q = 100 \text{ m}^3/\text{s}$

$$[y = 0.683 \text{ m}]$$

For kinematic wave model,

$$C_k = \frac{dQ}{dA}$$

$$= \frac{1}{b} \cdot \frac{dQ}{dy}$$

$$= \frac{1}{b} \cdot \frac{d}{dy} (188.56 y^{5/3})$$

$$= \frac{1}{b} \times 188.56 \times \frac{5}{3} y^{(5/3-1)}$$

$$= \frac{1}{40} \times 188.56 \times \frac{5}{3} \times y^{2/3}$$

$$= \frac{1}{40} \times 188.56 \times \frac{5}{3} \times 0.683^{2/3}$$

$$[\therefore C_k = 6.093 \text{ m/s}]$$

Courant condition,

$$\frac{C_k}{\Delta x / \Delta t} \leq 1$$

$$\text{on } \Delta t \leq \frac{\Delta x}{C}$$

$$\leq \frac{1500}{6.093}$$

$$\left[\begin{array}{l} \Delta t \leq 246.18 \text{ sec} \\ \Delta t \leq 4.1 \text{ min} \end{array} \right]$$

$\therefore$ Take time step: $\Delta t = 4 \text{ min}$

Q.2

Water flows through a rectangular channel 25m wide, having bed slope 0.015, $n = 0.035$. Take following flow rate $Q_i^{n+1} = 30 \text{ m}^3/\text{s}$, $Q_i^n = 22 \text{ m}^3/\text{s}$, $Q_{i+1}^n = 20 \text{ m}^3/\text{s}$

where, $i = \text{index for space}$, & $n = \text{index for time}$

taking, $\Delta x = 1500 \text{ m}$, and $\Delta t = 10 \text{ min}$. Determine Q_{i+1}^{n+1} using first order accurate linear kinematic wave model.

Assume lateral flow to be zero and $\left\{ \begin{array}{l} \text{wetted perimeter,} \\ \text{is equal to width of channel.} \end{array} \right.$

$\rightarrow$ SOL: F.W or B.W.

$$b = 25 \text{ m}$$

$$S = 0.015$$

$$n = 0.035$$

$$\Delta x = 1500 \text{ m}$$

$$\Delta t = 10 \text{ min}$$

Continuity eq:

$$\left[\frac{dQ}{dx} + \frac{dA}{dt} = 0 \right] \text{ u}$$

$$\left[\begin{array}{l} Q = 0 \\ P = b = 25 \text{ m} \end{array} \right]$$

For kinematic wave model,

$$[A = \alpha Q^\beta] \text{ --- (1)}$$

For $\alpha \& \beta$

Manning's eq:

$$Q = \frac{1}{n} A R^{2/3} S^{1/2}$$

$$Q = \frac{1}{n} \frac{A^{5/3}}{P^{2/3}} S^{1/2}$$

$$\therefore A = \left(\frac{n P^{2/3}}{\sqrt{S}} \right)^{3/5} \times Q^{3/5} \text{ --- (11)}$$

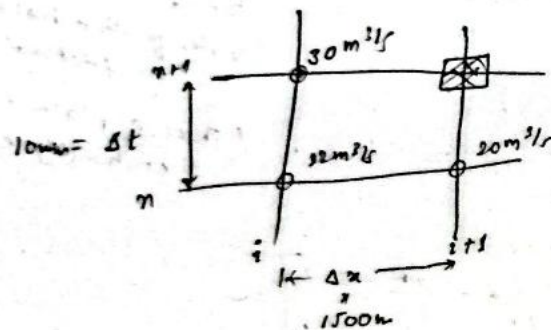
$S_0 = S_f$
(Gravity force balanced by frictional force)
(Thus explained by continuity eq)

Comparing (1) & (2)

$$\alpha = \left(\frac{n \cdot P^{2/3}}{\sqrt{S}} \right)^{3/5}$$

$$[P = 3/5 = 0.6]$$

$$\left[\alpha = \left(\frac{0.035 \times 25^{2/3}}{\sqrt{0.015}} \right)^{3/5} = 1.709 \right]$$



For KWM,

$$\frac{dQ}{d\alpha} + \frac{dA}{dt} = 0 \quad \text{--- (iii)}$$

$$A = \alpha Q^P \quad \text{--- (iv)}$$

$$\frac{dA}{dt} = \alpha P Q^{P-1} \frac{dQ}{dt} \quad \text{--- (v)}$$

$$\left[\frac{dQ}{d\alpha} + \alpha P Q^{P-1} \frac{dQ}{dt} = 0 \right] \quad \text{--- (vi)}$$

$$\frac{dQ}{d\alpha} = \frac{Q_{i+1}^{n+1} - Q_i^{n+1}}{\Delta\alpha}$$

$$\frac{dQ}{dt} = \frac{Q_{i+1}^{n+1} - Q_{i+1}^n}{\Delta t}$$

$$\text{Non-dervative: } Q = \frac{Q_{i+1}^n + Q_i^{n+1}}{2}$$

Thus,

substituting values

$$\frac{Q_{i+1}^{n+1} - Q_i^{n+1}}{\Delta\alpha} + \alpha P \left(\frac{Q_{i+1}^n + Q_i^{n+1}}{2} \right)^{P-1} \times \left(\frac{Q_{i+1}^{n+1} - Q_{i+1}^n}{\Delta t} \right) = 0$$

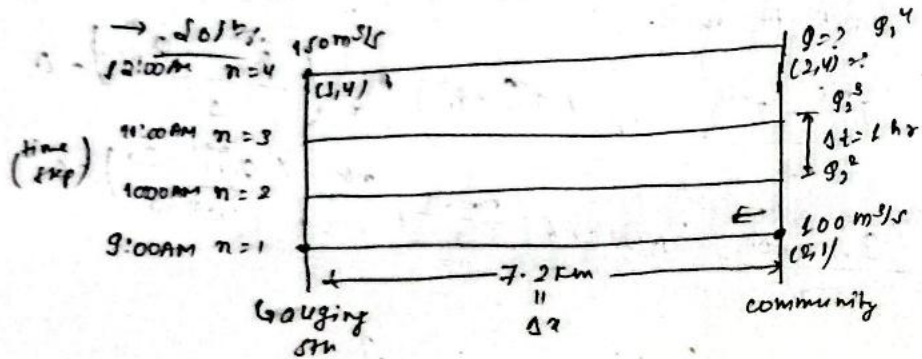
$$\frac{(Q_{i+1}^{n+1} - 30)}{1500} + 1.709 \times 0.6 \left(\frac{20 + 30}{2} \right)^{0.6-1} \times \left(\frac{Q_{i+1}^{n+1} - 20}{10 \times 60} \right) = 0$$

$$\Rightarrow [Q_{i+1}^{n+1} = 25.78 \text{ m}^3/\text{s}] \text{ u.}$$

Flood Routing

Q.5

A Flood of $150 \text{ m}^3/\text{s}$ peak discharge passed a gauging station at 12:00 noon on a river. There is a community adjacent to the river 7.2 km downstream. What will be the value of peak discharge at 12:00 noon if the velocity of flow is 1.2 m/s , peak discharge at community at 9:00 AM was $100 \text{ m}^3/\text{s}$. Use first order accurate scheme of kinematic wave function. Take $\Delta x = 7.2 \text{ km}$ and $\Delta t = 1 \text{ hr}$.



Nothing given so,
take, $Q_{1,1} = 100 \text{ m}^3/\text{s}$
 $Q_{1,2} = 100 \text{ m}^3/\text{s}$
 $Q_{1,3} = 100 \text{ m}^3/\text{s}$
 $Q_{1,4} = 100 \text{ m}^3/\text{s}$
KWM $\rightarrow \alpha \Delta x$

At community 9:00 AM
discharge, $Q = 100 \text{ m}^3/\text{s}$
velocity, $v = 1.2 \text{ m/s}$
Cont. eqn, $Q = A \times v$
 $A = \frac{Q}{v} = \frac{100}{1.2}$
 $[A = 83.33 \text{ m}^2]$

Kinematic wave function

$$A = \alpha Q^\beta$$

$$[\beta = 3/5 = 0.6] \text{ or } (\text{for all})$$

$$\therefore [\alpha = \frac{A}{Q^\beta} = \frac{83.33}{100^{0.6}} = 5.26]$$

Initial condition

$$[Q_1' = 100 \text{ m}^3/\text{s}]$$

Boundary condition

$$Q_1' = 100 \text{ m}^3/\text{s} = Q_1^2 = Q_1^3$$

1st order accurate kinematic wave model (B14)

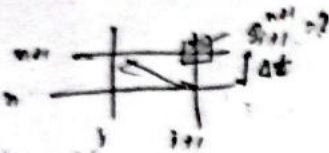
$$\frac{dQ}{dx} + \frac{dA}{dt} = 0$$

$$\frac{dQ}{dx} + \alpha \beta Q^{\beta-1} \frac{dQ}{dt} = 0$$

Step by step

12:00	4	150 m ³ /s	?
11:00	3	100 m ³ /s	?
10:00	2	100 m ³ /s	?
9:00	1	100 m ³ /s	100 m ³ /s
time (t)	n	Q (1)	7.2 km (2)
		↑ gauging stn	↑ community

Sum



$$\frac{dQ}{dn} + \frac{dA}{dt} = Q^0$$

$$\text{on } \frac{dQ}{dn} + \alpha f Q^{f-1} \frac{dQ}{dt} = 0$$

$$\text{or } \frac{Q_{i+1}^{n+1} - Q_i^{n+1}}{\Delta n} + \alpha f \left(\frac{Q_{i+1}^n + Q_i^n}{2} \right) \left(\frac{Q_{i+1}^{n+1} - Q_i^{n+1}}{\Delta t} \right) = 0$$

$$\text{or } Q_{i+1}^{n+1} = \frac{\frac{\Delta t}{\Delta n} Q_i^{n+1} + \alpha f Q_{i+1}^n \left(\frac{Q_{i+1}^n + Q_i^n}{2} \right)^{f-1}}{\frac{\Delta t}{\Delta n} + \alpha f \left(\frac{Q_{i+1}^n + Q_i^n}{2} \right)^{f-1}}$$

$$\rightarrow \begin{cases} \Delta t = 1 \text{ hr} \\ \Delta n = 7.2 \text{ km} \\ \frac{\Delta t}{\Delta n} = \frac{1 \times 60 \times 60}{7.2 \times 1000} = 0.5 \end{cases}$$

$$\rightarrow \begin{cases} \alpha = 5.26 \\ f = 0.6 \end{cases} \alpha f = 3.156$$

$$\rightarrow f-1 = -0.4$$

For Q_2^2

$$Q_2^2 = \frac{0.5 \times 100 + 3.156 \times 100 \left(\frac{100+100}{2} \right)^{-0.4}}{0.5 + 3.156 \left(\frac{100+100}{2} \right)^{-0.4}}$$

$$[Q_2^2 = 100 \text{ m}^3/\text{s}]$$

For Q_2^3

$$Q_2^3 = 100 \text{ m}^3/\text{s} \quad (\text{At } 11:00 \text{ AM community})$$

For Q_2^4

$$Q_2^4 = 0.5 \times 150$$

		150	?
n=4			
		100	100
n=3			
		100	100
n=2			
		100	100
n=1			
	i=3		2
	Gauging		community

$$Q_2^4 = \frac{0.5 \times 150 + 3.156 \times 100 \left(\frac{150+100}{2} \right)^{-0.4}}{0.5 + 3.156 \left(\frac{150+100}{2} \right)^{-0.4}}$$

$$[Q_2^4 = 126.11 \text{ m}^3/\text{s}] \rightarrow \text{community discharge at } 12:00 \text{ AM}$$

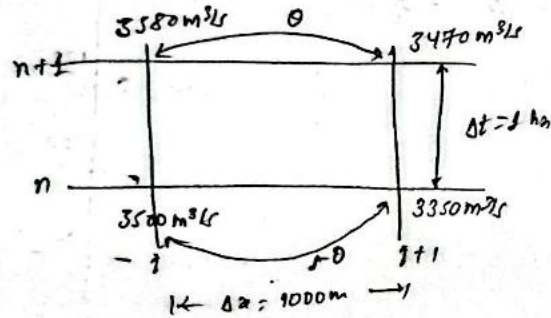
9.4

The value of flow rate at four points in the space time grid are shown in figure below. $\Delta t = 1 \text{ hr}$.

$\Delta x = 1000 \text{ m}$, $\theta = 0.55$.

Calculate value of $\frac{\partial Q}{\partial t}$ and $\frac{\partial Q}{\partial x}$ by four points

Implicit method, $\theta =$ weighing factor.



→ Solⁿ

$$\frac{dQ}{dt} = \frac{1}{2} \left[\left(\frac{Q_{i+1}^{n+1} - Q_{i+1}^n}{\Delta t} \right) + \left(\frac{Q_i^{n+1} - Q_i^n}{\Delta t} \right) \right]$$

$$= \frac{1}{2} \left[\left(\frac{3580 - 3500}{1 \times 60 \times 60} \right) + \left(\frac{3470 - 3350}{1 \times 60 \times 60} \right) \right]$$

$$\left(\frac{dQ}{dt} = 0.028 \text{ m}^3/\text{s}/\text{s} \right)$$

$$\frac{dQ}{dx} = \theta \left(\frac{Q_{i+1}^{n+1} - Q_i^{n+1}}{\Delta x} \right) + (1-\theta) \left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x} \right)$$

$$= 0.55 \left(\frac{3470 - 3580}{1000} \right) + (1-0.55) \left(\frac{3350 - 3500}{1000} \right)$$

$$\left[\frac{dQ}{dx} = -0.128 \text{ m}^3/\text{s}/\text{m} \right]$$

Dynamic wave model

9.5

Water flows in a rect. channel of width 25m having bed slope 0.015. Assume no lateral inflow. Consider wetted perimeter is equal to width of channel. Taking

$\Delta x = 1500 \text{ m}$, $\Delta t = 10 \text{ min}$ and $n = 0.035$ Give

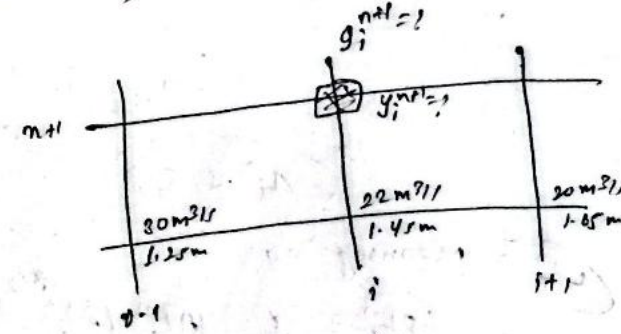
$Q_{i-1}^n = 30 \text{ m}^3/\text{s}$, $Q_i^n = 22 \text{ m}^3/\text{s}$, $Q_{i+1}^n = 20 \text{ m}^3/\text{s}$

$y_{i-1}^n = 1.25 \text{ m}$, $y_i^n = 1.45 \text{ m}$, $y_{i+1}^n = 1.65 \text{ m}$

Using an explicit method, determine the depth and discharge at the grid at $(i, n+1)$ for full Saint Venant equation (i.e. Dynamic wave Model)

→ Give

$b = 25 \text{ m}$
 $s = 0.015$
 $q = 0$
 $P = b = 25 \text{ m}$
 $\Delta x = 1500 \text{ m}$
 $\Delta t = 10 \text{ min}$
 $n = 0.035$



Area (A) = $b \times y$

$$A_{i-1}^n = b y_{i-1}^n = 25 \times 1.25 = 31.25 \text{ m}^2$$

$$A_i^n = b y_i^n = 25 \times 1.45 = 36.25 \text{ m}^2$$

$$A_{i+1}^n = b y_{i+1}^n = 25 \times 1.65 = 41.25 \text{ m}^2$$

Using continuity eqⁿ:

$$\frac{dQ}{dx} + \frac{dA}{dt} = q$$

$$\frac{Q_{i+1}^n - Q_i^n}{\Delta x} + \frac{A_{i+1}^{n+1} - A_i^n}{\Delta t} = 0$$

$$\frac{20 - 22}{1500} + \frac{A_{i+1}^{n+1} - 36.25}{10 \times 60} = 0$$

$$\Rightarrow [A_{i+1}^{n+1} = 37.05 \text{ m}^2]$$

explicit
for $t = n+1$
use $t = n$ values

$$LC \left[y_i^{n+1} = \frac{A_i^{n+1}}{b} = \frac{37.05}{25} = 1.482m \right]$$

For Full Saint-Venant eq:

$$\frac{dQ}{dt} + \frac{\partial}{\partial x} \left(\frac{Q^2}{A} \right) + gA \frac{dy}{dx} - gA (S_0 - S_f) = 0$$

$(A \& S_f) \rightarrow$ Non-derivative term

For Non-derivative term

$$f = f_i^n \text{ or } \frac{f_{i-1}^n + f_{i+1}^n}{2}$$

$$\text{or } \frac{f_{i-1}^n + f_i^n + f_{i+1}^n}{3}$$

$$A_i^n \& S_{fi}^n$$

(LC) Manning's eq:

$$(Q)_i^n = \frac{1}{n} (A)_i^n (R)_i^n (S_{fi})_i^n^{1/2}$$

$$\text{or } 22 = \frac{1}{0.035} \times 36.25 \times \left(\frac{36.25}{25} \right)^{2/3} \times (S_{fi})_i^n^{1/2}$$

$$\Rightarrow [(S_{fi})_i^n = 2.146 \times 10^{-4}]$$

Thus

$$\frac{(Q_i^{n+1}) - Q_i^n}{\Delta t} + \frac{1}{\Delta x} \left[\left(\frac{Q^2}{A} \right)_{i+1} - \left(\frac{Q^2}{A} \right)_i \right]$$

$$+ gA_i^n \left(\frac{y_{i+1}^n - y_i^n}{\Delta x} \right) - gA_i^n (S_0 - S_{fi}) = 0$$

$$\rightarrow [Q_i^{n+1} = 3149.74 m^3/s]$$

Q.6 Non-linear KWM

Use finite difference non-linear scheme, compute discharge at 1km downstream of location x at time 14:00 hr. for following set of data

Rectangular channel width 20m

Bed slope = 0.001

Manning's n = 0.03

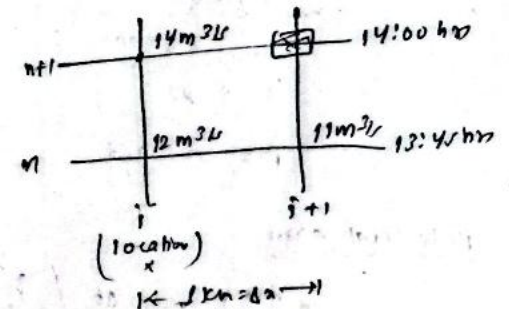
Discharge at location x at time 14:00 hrs = 14 m^{3/s}

" " " " " 13:45 hrs = 12 m^{3/s}

" " 1 km downstream of location x at time 13:45 hrs = 11 m^{3/s}

Assume: no lateral flow & wetted perimeter equal to width of channel.

$$\text{Let } \left[\begin{array}{l} b = 20m \\ s = 0.001 \\ n = 0.03 \\ P = b = 20m \end{array} \right]$$



For non-linear scheme,

$$\frac{dQ}{dx} + \frac{dP}{dt} = 0$$

$$\Rightarrow \left(\frac{Q_{i+1}^{n+1} - Q_i^{n+1}}{\Delta x} \right) + \left(\frac{A_{i+1}^{n+1} - A_i^{n+1}}{\Delta t} \right) = 0$$

For Kinematic model: $A = \alpha Q^f$

$$A_{i+1}^{n+1} = \alpha (g_{i+1}^{n+1})^\beta$$

$$(A_{i+1}^n) = \alpha (g_{i+1}^n)^\beta$$

$$\Delta t \times \frac{(g_{i+1}^{n+1}) - g_{i+1}^n}{\Delta t} + \frac{\alpha [(g_{i+1}^{n+1})^\beta - (g_{i+1}^n)^\beta]}{\Delta t} \times \Delta t = 0$$

$$\Delta t \times \frac{g_{i+1}^{n+1} - g_{i+1}^n}{\Delta t} + \alpha (g_{i+1}^{n+1})^\beta = \frac{\Delta t}{\Delta t} g_{i+1}^n + \alpha (g_{i+1}^n)^\beta$$

Manning's eqn

$$\alpha = \left(\frac{n P^{2/3}}{\sqrt{S}} \right)^{3/5}$$

$$= \left(\frac{0.03 \times 20^{2/3}}{\sqrt{0.001}} \right)^{3/5}$$

$$= 3.21$$

$$\beta = 3/5 = 0.6$$

Residual eqn

$$F(g_{i+1}^{n+1}) = \frac{\Delta t}{\Delta t} (g_{i+1}^{n+1}) + \alpha (g_{i+1}^{n+1})^\beta - \frac{\Delta t}{\Delta t} (g_{i+1}^n) - \alpha (g_{i+1}^n)^\beta$$

$$\text{or } F(g_{i+1}^{n+1}) = \frac{15 \times 60}{1000} (g_{i+1}^{n+1}) + 3.21 (g_{i+1}^{n+1})^{0.6} - \frac{15 \times 60}{1000} (14) - 3.21 \times (11)^{0.6}$$

$$F(g_{i+1}^{n+1}) = 0.9 g_{i+1}^{n+1} + 3.21 (g_{i+1}^{n+1})^{0.6} - 26.13 \quad \text{--- (1)}$$

Newton's Raphson method

$$(g_{i+1}^{n+1})_{k+1} = (g_{i+1}^{n+1})_k - \frac{F(g_{i+1}^{n+1})_k}{F'(g_{i+1}^{n+1})_k} \rightarrow g_{i+1}^{n+1}$$

$$F'(g_{i+1}^{n+1}) = 0.9 + 1.926 (g_{i+1}^{n+1})^{-0.4} \quad \text{--- (11)}$$

For 1st iteration

$$g_{i+1}^{n+1} = 14 \text{ m/s}$$

Iteration no.	$(g_{i+1}^{n+1})_k$	$F(g_{i+1}^{n+1})$	$F'(g_{i+1}^{n+1})$	$(g_{i+1}^{n+1})_{k+1}$
1	14	2.108	1.570	12.657
2	12.657	2.108 -0.0188	1.598	12.669
3	12.669	9.258×10^{-4}	1.598	12.669

Thus

$$(g_{i+1}^{n+1})_k = (g_{i+1}^{n+1})_{k+1}$$

Thus

$$[g_{i+1}^{n+1} = 12.669 \text{ m/s}]$$

Ch. 6

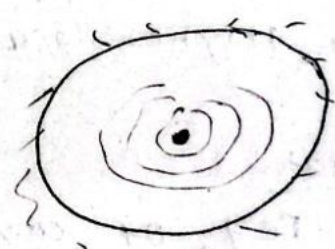
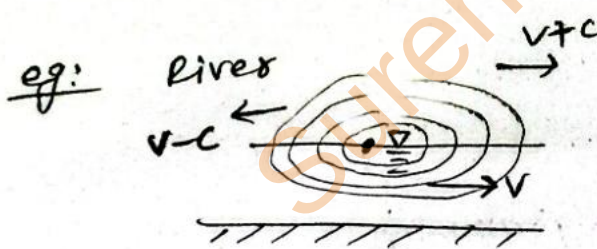
Method of Characteristics (MOC)

Prev. $\left. \begin{array}{l} \text{PDE} \rightarrow \text{Numerical solution} \\ \text{FDM} \rightarrow \text{tool to solve PDE} \end{array} \right\}$
discretization

MOC $\rightarrow$ also a tool of solving PDE by converting them to ordinary diff^l eq^s (ODE) and solving the ODEs by numerical or graphical technique

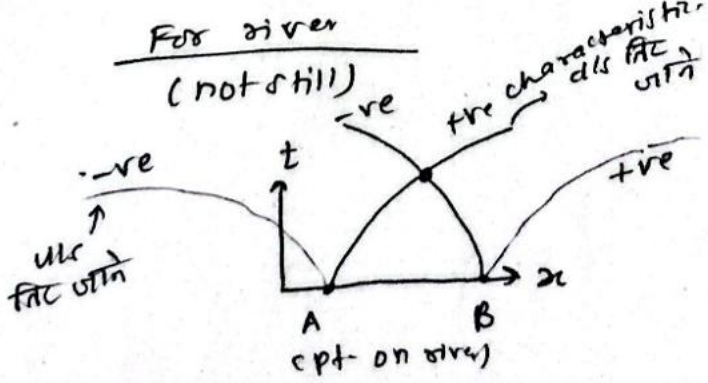
Characteristics:

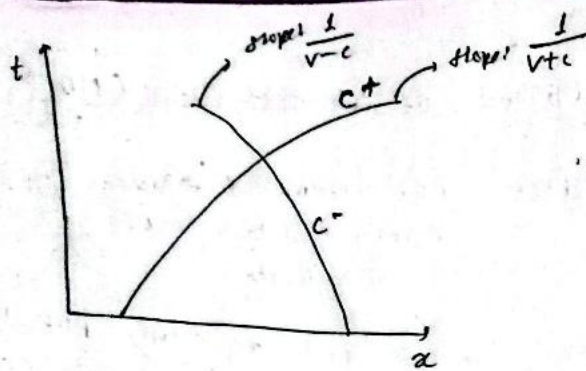
- curve for plot of $\frac{dx}{dt} = V \pm c$ in $x-t$ plane
- $V \pm c$: velocity of propagation of disturbance
- slope of wave, $\tan \theta = \frac{dt}{dx} = \frac{1}{V \pm c}$
- slope of characteristic line: $\frac{1}{V+c}$ along C^+ and $\frac{1}{V-c}$ along C^-



Still Pond

wave travel in radially outward dirⁿ.
(travel all side equally,





6.2 Application of MOC to unsteady pipe flow prob

Momentum eqⁿ:-

$$L_1 = \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} + g \sin \theta + \frac{f V |V|}{2D} = 0 \quad (i)$$

continuity eqⁿ:-

$$L_2 = \frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho c^2 \frac{\partial v}{\partial x} = 0 \quad (ii)$$

Requirements for solution

- Initial condition: value of variables at $t=0$
- Boundary conditions (BC): conditions at the boundary of physical system

Adv of MOC

- ↳ Handling of complex system
- ↳ Easy to establish stability of solⁿ
- ↳ Easy to handle and program BC
- ↳ Easy to program
- ↳ Higher accuracy

Limitations

- ↳ Req. of computer programming.

V & $P \rightarrow$ unknown.

$v \rightarrow$ Avg. velocity over section
 $c \rightarrow$ velocity of wave = $\sqrt{\frac{K/\rho}{1 + \frac{Dk}{EF}}}$

$\sin \theta \rightarrow$ slope of pipe
 $f \rightarrow$ friction factor.
 $P \rightarrow$ press at point
 $\rho \rightarrow$ density of fluid.

Combining two eqⁿs.

$$L = L_1 + \lambda L_2 = 0, \quad \lambda = \text{multiplier}$$

$$L = \left[\frac{\partial v}{\partial t} + v + \lambda \rho c^2 \right] + \lambda \left[\frac{\partial p}{\partial x} \left(\frac{1}{\rho} + v \right) + \frac{\partial p}{\partial t} \right] +$$

$$g \sin \theta + \frac{f V |V|}{2D} = 0 \quad (iii)$$

we have

$$\frac{dv}{dt} = \frac{\partial v}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial v}{\partial t} \quad \& \quad \frac{dp}{dt} = \frac{\partial p}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial p}{\partial t}$$

(PDE $\rightarrow$ ODE)

substituting:

$$L = \frac{dv}{dt} + \lambda \frac{dp}{dt} + g \sin \theta + \frac{f V |V|}{2D} = 0 \quad (iv)$$

Above expression is valid

If $\frac{da}{dt} = (v + \lambda sc^2)$ for velocity post.

$\frac{da}{dt} = \left(\frac{1}{sc} + v\right)$ for pressure post.

Equating,

$$(v + \lambda sc^2) = \frac{1}{sc} + v = \frac{da}{dt}$$

$$\Rightarrow \left[\lambda = \pm \frac{1}{sc}\right]$$

$$\left\{ \frac{da}{dt} = v \pm c \right\}$$

Characteristic eqs.

→ Substituting in eqⁿ (iv)

$$L = \frac{dv}{dt} \pm \frac{1}{sc} \frac{dp}{dt} + g \sin \theta + \frac{f|v|v}{2D} = 0 \quad \text{--- (v)}$$

with, $\lambda = \pm \frac{1}{sc}$, $\frac{da}{dt}$ becomes

$$\frac{da}{dt} = v \pm c \quad \text{--- (vi)}$$

→ Eqⁿ (v) is valid if eqⁿ (vi) is satisfied.

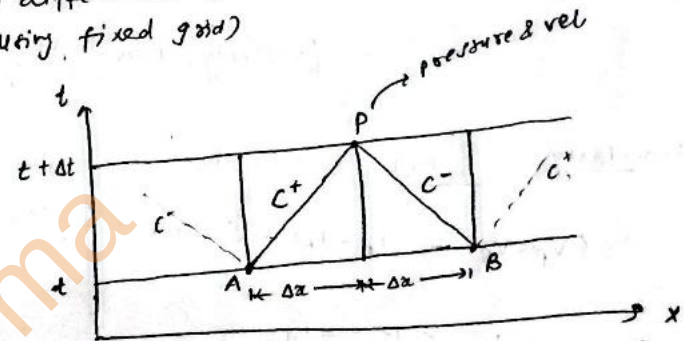
→ Eqⁿ (v) and (vi) are characteristic equations in ODE which can be solved by FDM,

→ $v \frac{\partial v}{\partial x}$ in eqⁿ (i) and $v \frac{\partial p}{\partial x}$ in eqⁿ (ii) are much smaller than $\frac{\partial v}{\partial t}$ and $\frac{\partial p}{\partial t}$. So, they can also be neglected and, $\frac{da}{dt} = \pm c$.

6.2.1 Finite difference solutions of characteristic eqⁿ

Chr. eqⁿ converted to F.D eqⁿ by integration

a. Finite difference eqⁿ in terms of velocity and pressure (using fixed grid)



Given or known: v_A, p_A, x_A, t_A & v_B, p_B, x_B, t_B
(vel. & press. at prev. time levels)

$$\text{slope of AP} = \frac{dt}{dx} = \frac{1}{c}$$

$$\text{slope of BP} = \frac{dt}{dx} = -\frac{1}{c}$$

P: pt. of intersection.

Taking C+

$$\frac{dv}{dt} + \frac{1}{sc} \frac{dp}{dt} + g \sin \theta + \frac{f|v|v}{2D} = 0 \quad (\text{eqⁿ (v)})$$

$$\text{Multiplying throughout by } scdt \text{ and integrating from A to P,}$$

$$sc \int_A^P dv + \int_A^P dp + \int_A^P sc g \sin \theta dt + \int_A^P sc \frac{f|v|v}{2D} dt = 0$$

as $\frac{da}{dt} = c$, replace, cdt by dx ,

$$sc \int_A^P dv + \int_A^P dp + \int_A^P sc g \sin \theta dx + \int_A^P sc \frac{f|v|v}{2D} dx = 0$$

Writing in finite difference form.

$$\rho C (V_P - V_A) + (P_P - P_A) + \rho g \sin \theta \Delta x + \frac{\rho \Delta x f V_A |V_A|}{2D} = 0 \quad \text{--- (VII)}$$

[velocity: known pt. i.e. A at t & $t + \Delta t$]

Similarly, B to P (C⁻ line)

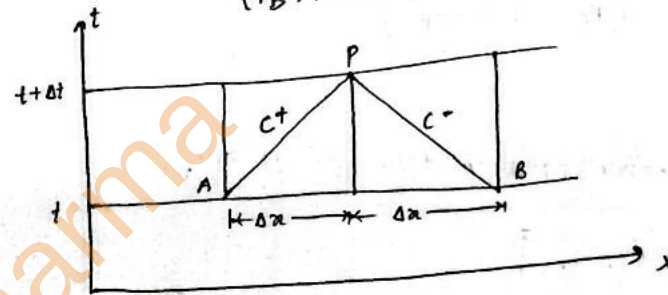
$$\rho C (V_P - V_B) - (P_P - P_B) + \rho g \sin \theta \Delta x + \frac{\rho \Delta x f V_B |V_B|}{2D} = 0 \quad \text{--- (VIII)}$$

as, $C dt = -dx$

& $\int_B^P -dx = -(x_P - x_B) = -(-\Delta x) = \Delta x$

Eqⁿ (VII) & (VIII) are simultaneously solved to obtain P_P & V_P

Q. Following data are given at two pts A and B along a pipe of dia 80cm carrying water: $V_A = 5 \text{ m/s}$, $P_A = 100 \text{ kPa}$, $\Delta x = 100 \text{ m}$, $\Delta t = 10 \text{ sec}$, $f = 0.02$, $C = 1200 \text{ m/s}$, elev. diff. for 100m distance = 1m. Using finite diff. form of characteristic eqⁿ. Compute vel. and pres. at point P.
 (Slope fraction)



→ Solⁿ.

$\tan \theta = \frac{L}{100}$, for small θ , $\sin \theta \approx \tan \theta = \frac{L}{100}$

C⁺ line

$$\rho C (V_P - V_A) + (P_P - P_A) + \rho g \sin \theta \Delta x + \frac{\rho f \Delta x V_A |V_A|}{2D} = 0$$

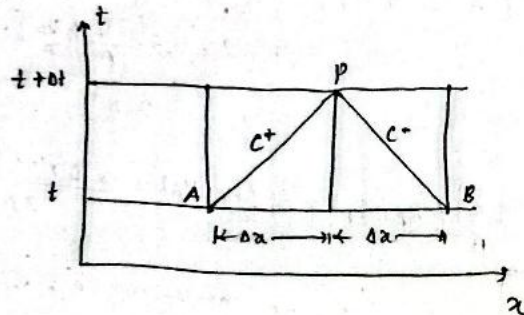
or, $1000 \times 1200 (V_P - 5) + (P_P - 100 \times 10^3) + 9.81 \times 1000 \times \frac{1}{100} \times 100 + 1000 \times 0.02 \times 100 \times \frac{5 \times 5}{2 \times 0.3} = 0 \quad \text{--- (I)}$

Similarly (C⁻ line)

$$\rho C (V_P - V_B) - (P_P - P_B) + \rho g \sin \theta \Delta x + \frac{\rho f \Delta x V_B |V_B|}{2D} = 0$$

or, $1000 \times 1200 (V_P - 5.1) - (P_P - 120 \times 10^3) + 1000 \times 9.81 \times \frac{1}{100} \times 100 + 1000 \times 0.02 \times 100 \times \frac{5.1 \times 5.1}{2 \times 0.3} = 0 \quad \text{--- (II)}$

Q2 Following data are given at two points A and B along a pipe of diameter 20cm carrying water. $Q_A = 0.3 \text{ m}^3/\text{s}$, $Q_B = 0.33 \text{ m}^3/\text{s}$, $H_A = 20.5 \text{ m}$, $H_B = 21 \text{ m}$, $\Delta x = 100 \text{ m}$, $\Delta t = 10 \text{ sec}$, $f = 0.02$, $c = 1200 \text{ m/s}$, elev diff for 100m distance = 1m. Using finite difference form of the eqn, compute discharge and head at point P.



→ Soln.

$$C^+ \text{ line: } H_P = H_A - B(Q_P - Q_A) - R Q_A | Q_A |$$

$$C^- \text{ line: } H_P = H_B + B(Q_P - Q_B) + R Q_B | Q_B |$$

$$B = \frac{c}{gA} = \frac{1200}{9.81 \times \pi \times \frac{0.2^2}{4}} = 3893.7$$

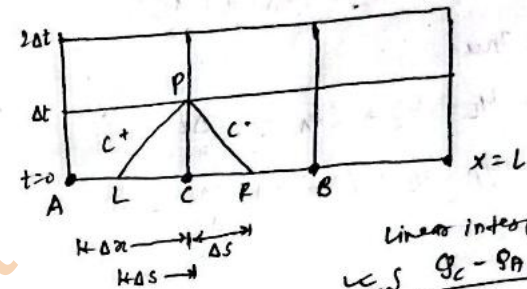
$$R = \frac{f \Delta x}{2gDA^2} = \frac{0.02 \times 100}{2 \times 9.81 \times 0.2 \times \left(\pi \times \frac{0.2^2}{4}\right)^2} = 516.42$$

$$\text{eqn (i)} \\ H_P = 20.5 - 3893.7 (Q_P - 0.3) - 516.42 \times 0.3 \times 10.3 \quad \text{--- (i)}$$

$$\text{eqn (ii)} \\ H_P = 21 + 3893.7 (Q_P - 0.33) + 516.42 \times 0.33 \times 10.33 \quad \text{--- (ii)}$$

Solving, $H_P = 20.77 \text{ m}$
 $Q_P = 0.302 \text{ m}^3/\text{s}$

Solution technique for water hammer problem using specified time intervals (characteristics not on diagonal, interpolating value, time intervals known).



→ (Interpolating value at L & R)

$$\text{For } C^+: H_P = H_L - B(Q_P - Q_L) - R Q_L | Q_L | \quad \text{--- (a)}$$

$$\text{For } C^-: H_P = H_R + B(Q_P - Q_R) + R Q_R | Q_R | \quad \text{--- (b)}$$

$$\text{Coordinate of L (x}_L\text{)} = x_P - \Delta s = x_P - c \Delta t = x_P - c(t_P - t_L)$$

$$\text{Coordinate of R (x}_R\text{)} = x_P + \Delta s = x_P + c \Delta t = x_P + c(t_P - t_R)$$

$$\rightarrow \frac{\Delta s}{\Delta t} = V_L + c \quad \text{--- (iii) [from } \frac{dx}{dt} = V + c \text{]}$$

V_L is much smaller than $c \rightarrow$ neglect.

$$\frac{\Delta s}{\Delta t} = c$$

→ Thus,

$$Q_L = Q_C + \frac{c \Delta t}{\Delta x} (Q_A - Q_C) \quad \text{--- (iv)}$$

Similarly,

$$Q_R = Q_C + \frac{c \Delta t}{\Delta x} (Q_B - Q_C) \quad \text{--- (v)}$$

Similarly for head,

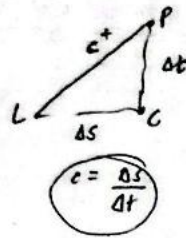
$$\frac{H_L - H_C}{H_P - H_C} = \frac{\Delta S}{\Delta x}$$

$$H_L = H_C + (H_P - H_C) \frac{\Delta S}{\Delta x}$$

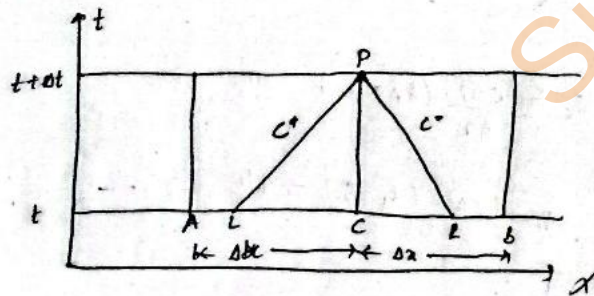
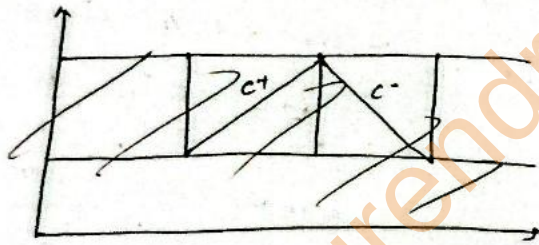
Thus,

$$H_L = H_C + (H_A - H_C) \frac{C \Delta t}{\Delta x} \quad \text{--- (V)}$$

$$H_P = H_C + (H_B - H_C) \frac{C \Delta t}{\Delta x} \quad \text{--- (W)}$$



Q.3. Following data are given at two points A and B along a pipe of diameter 40cm carrying water: $Q_A = 0.5 \text{ m}^3/\text{s}$, $Q_B = 0.55 \text{ m}^3/\text{s}$, $Q_C = 0.53 \text{ m}^3/\text{s}$, $H_A = 25 \text{ m}$, $H_B = 25.5 \text{ m}$, $H_C = 25.3 \text{ m}$, $\Delta x = 1500 \text{ m}$, $\Delta t = 1 \text{ sec}$, $f = 0.02$, $c = 1200 \text{ m/s}$. elev. diff. for 100m distance = 1m. Using finite difference form of continuity compute discharge and head at point P.



We interpolate L & E value,

$$\frac{\Delta S}{\Delta t} = C$$

$$\Delta S = 1200 \times 1 = 1200 \text{ m}$$

$$Q_L = Q_C + (Q_A - Q_C) \frac{C \Delta t}{\Delta x}$$

$$= 0.53 + (0.5 - 0.53) \times \frac{1200 \times 1}{1500}$$

$$= 0.506 \text{ m}^3/\text{s}$$

$$Q_E = Q_C + (Q_B - Q_C) \frac{C \Delta t}{\Delta x}$$

$$= 0.53 + (0.55 - 0.53) \times \frac{1200 \times 1}{1500}$$

$$= 0.546 \text{ m}^3/\text{s}$$

$$H_L = 25.3 + (25 - 25.3) \frac{1200 \times 1}{1500} = 25.06 \text{ m}$$

$$H_E = 25.3 + (25.5 - 25.3) \frac{1200 \times 1}{1500} = 25.46 \text{ m}$$

Now,

$$\text{For } C^+L: H_P = H_L - B(Q_P - Q_L) - R Q_L(Q_L)$$

$$\text{For } C^-: H_P = H_E + B(Q_P - Q_E) + R Q_E(Q_E)$$

$$B = \frac{c}{gA} = \frac{1200}{9.81 \times \frac{\pi \times 0.4^2}{4}} = 973.42$$

$$R = \frac{f \Delta x}{2gDA^3} = \frac{0.02 \times 1500}{2 \times 9.81 \times 0.4 \times \left(\frac{\pi \times 0.4^2}{4}\right)^2} = 242.07$$

$$H_P = 25.06 - 973.42(Q_P - 0.506) - 16.15 \times 0.506 \times 0.506 \quad \text{--- (1)}$$

$$H_P = 25.46 + 973.42(Q_P - 0.546) + 16.15 \times 0.546 \times 0.546 \quad \text{--- (2)}$$

$$\text{Solving, } H_P = 5.13 \text{ m}$$

$$Q_P = 0.521 \text{ m}^3/\text{s}$$

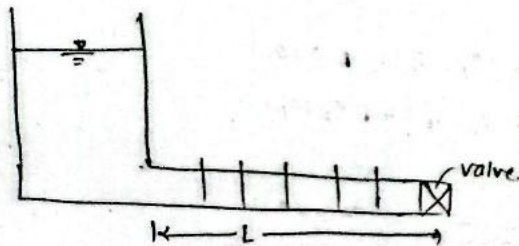
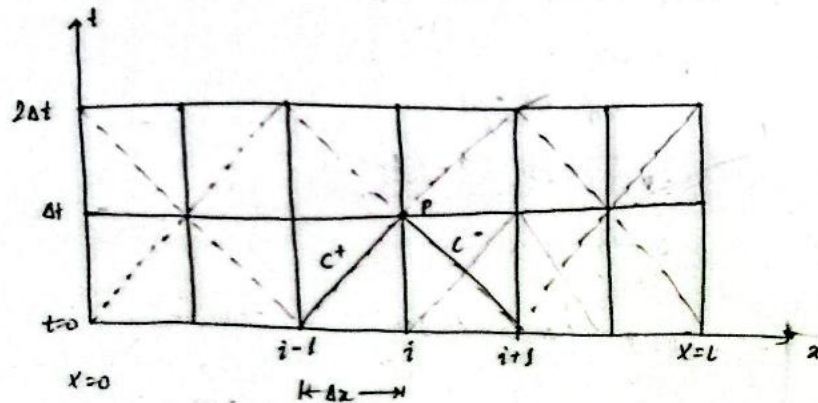
= Algorithm for solving water hammer problem (using rectangular grid, characteristic paths for diagonals)

We have,

$$\text{For } C^+ : H_p = H_p - B(\phi_p - \phi_p) - R\phi_p|\phi_p| \quad \text{--- (A)}$$

$$\text{For } C^- : H_p = H_p + B(\phi_p - \phi_p) + R\phi_p|\phi_p| \quad \text{--- (B)}$$

Find, H & ϕ at grid pt. i using rectangular grid.



divided into 10 of reach.

Say, $L = 1000m$

No. of reach = 10

$$\therefore \Delta x = \frac{1000}{10} = \frac{L}{N} = 100m$$

→ Conditions at time, $t=0$ i.e. initial condition must be given.

$$\text{Take: } \Delta x = L/N \quad \& \quad \Delta t = \Delta x/C$$

→ L = length of pipe, N = No. of reach,

→ Constructed rectangular grid in such a way that C^+ and C^- are the diagonals of the grid. This approach is suitable for single pipe line.

→ Known: $\phi_{i-1}, H_{i-1}, \phi_{i+1}, H_{i+1}$ (from initial condition or from previous time step computation)

→ To compute: H_p, ϕ_p

→ From eq. (A) (replacing A by $i-1$)

$$H_p = H_{i-1} - B(\phi_p - \phi_{i-1}) - R\phi_{i-1}|\phi_{i-1}|$$

$$\text{or } H_p + B\phi_p = H_{i-1} + \phi_{i-1}(B - R|\phi_{i-1}|) \quad \text{--- (C)}$$

$$\text{or } H_p + B\phi_p = C_p \text{ along } C^+ \quad \text{--- (C)}$$

$$\text{where, } C_p = H_{i-1} + \phi_{i-1}(B - R|\phi_{i-1}|) = \text{Known term.}$$

→ From eq. (B) (replacing B by $i+1$)

$$H_p = H_{i+1} + B(\phi_p - \phi_{i+1}) + R\phi_{i+1}|\phi_{i+1}|$$

$$\text{or } H_p - B\phi_p = C_M \text{ along } C^- \quad \text{--- (D)}$$

$$\text{where, } C_M = H_{i+1} - \phi_{i+1}(B + R|\phi_{i+1}|) = \text{Known term.}$$

→ Solving eq. (C) & (D)

$$H_p = \frac{1}{2}(C_p - C_M) \quad \text{--- (E)}$$

$$\phi_p = \frac{1}{B}(C_p - H_p) \quad \text{--- (F)}$$

Algorithm

compute

(a) Take $\Delta x = L/N$ and $\Delta t = \Delta x/c$

(b) compute B and K ($B = \frac{c}{gA}$ and $K = \frac{f \Delta x}{8gDA^3}$)

(c) For time t,

Res. good pt. i, compute C_p and C_H .

compute H_{Pi} and Q_{Pi}

write eqn. (a), (b), (c), (d)

(d) Perform similar computations for all interior nodes.

(e) Advance to next time step.

Stability of MOC solution

Courant number (C_n) ≤ 1

$C_n = \frac{1V+c}{\Delta x/\Delta t} = \frac{\text{theoretical vel.}}{\text{numerical vel.}}$

For stability,

$C_n \leq 1$

or $\frac{1V+c}{\Delta x/\Delta t} \leq 1$

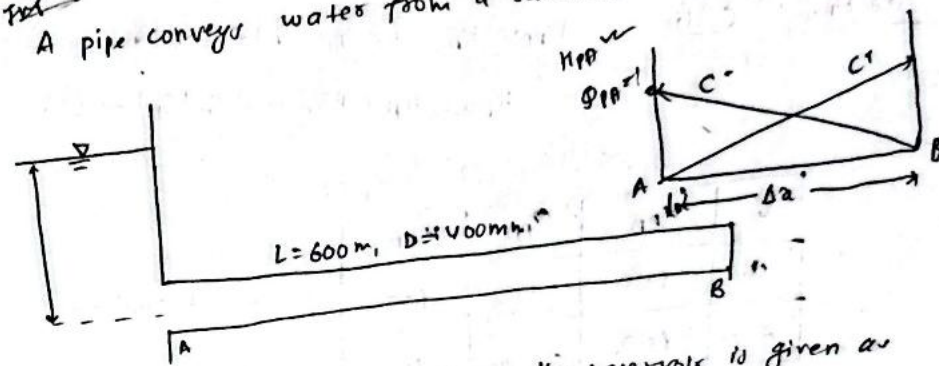
or $\Delta t \leq \frac{\Delta x}{1V+c}$

Divide pipe into N segments and compute Δt for all segments

$\Delta t_k \leq \frac{L}{N(1V+c)}$

Take minimum value of Δt for stability

4. A pipe conveys water from a reservoir as shown in the figure.



Take $f = 0.02$, $c = 1100 \text{ m/s}$. The HGL at the reservoir is given as $H_{PP} = 100 + 2 \sin(\pi t)$. The discharge at the dist. end is zero at all times. By using only one reach, compute discharge from A and elev. of HGL at B at 3 sec using discretized eqn of the MOC in the form of HGL and discharge.

→ Soln,

$Q_B = 0$

$N = 1$

$L = 600$

$\Delta x = \frac{L}{N} = 600 \text{ m}$

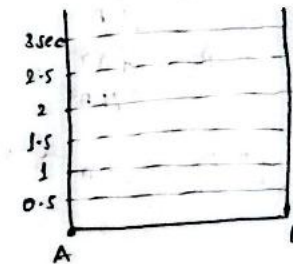
$\Delta t = \frac{\Delta x}{c} = \frac{600}{1100} = 0.5 \text{ sec}$

$t = 3 \text{ sec} \Rightarrow 6 \text{ steps}$

$H_{PP} = 100 + 2 \sin(\pi t) \quad \text{--- (1)}$

For Q_{PP} , C-line solve $\frac{dH}{dt} = 0$ as C-line intersects at P

C-line: $H_{PP} = H_B + B(Q_{PP} - Q_B) + R(Q_B | Q_B)$
 $\Rightarrow Q_{PP} = (H_{PP} - H_B) / B \quad \text{--- (2)}$



for H_{PB} , C+line: $H_{PB} = H_A - B(\phi_{PB} - \phi_A) - R\phi_A|\phi_A|$
 $H_{PB} = H_A + 8\phi_A - R\phi_A|\phi_A|$ — (iii)

t	H_{PA} $\xrightarrow{\text{eq}^{\text{II}}}$	ϕ_{PA} $\xrightarrow{\text{eq}^{\text{II}}}$	H_{PB} $\xrightarrow{\text{eq}^{\text{III}}}$	
0	100	0	100	← Initial condition
0.5	102	0.00205	100	
1	100	0	104	$\phi_A = 0.00205$ $H_A = 102$
1.5	99.98	-0.0062	100	
2.0	100	0	99.92	$\phi_A = -0.0062$ $H_A = 99$
2.5	99.102	0.0103	100	
3.0	100	0	99.2	$\phi_A = 0.0103$ $H_A = 102$

eq^{II}: $H_{PA} \rightarrow$ current
 $H_B \rightarrow$ next step at $t+\Delta t$

$[t=0.5: H_{PA}=102, H_B=100]$

$$B = \frac{c}{gA} = 973.42$$

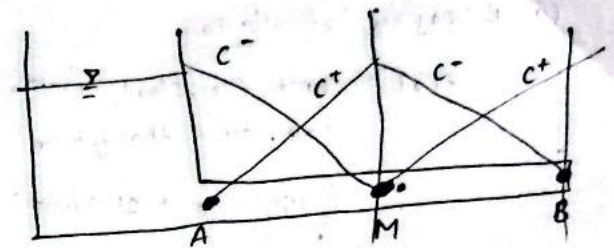
$\{t \text{ after next current time}\}$

$$R = \frac{f \Delta x}{2gD^5} = 98.62$$

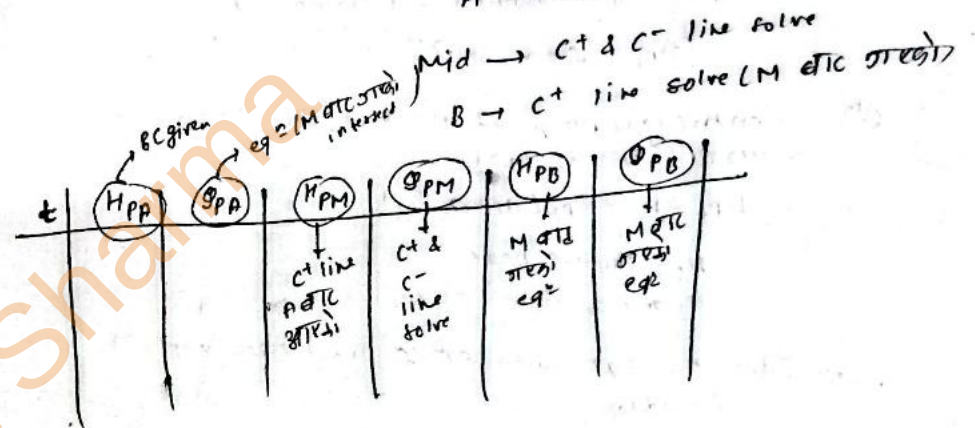
eq^{III}, $H_A \rightarrow$ after Δt time Δt
 $[t=0.5; H_A=100, \phi_A=0]$

$B \rightarrow$ etc head vary
 $\rightarrow$ discharge $\rightarrow$ pipe to reservoir flow

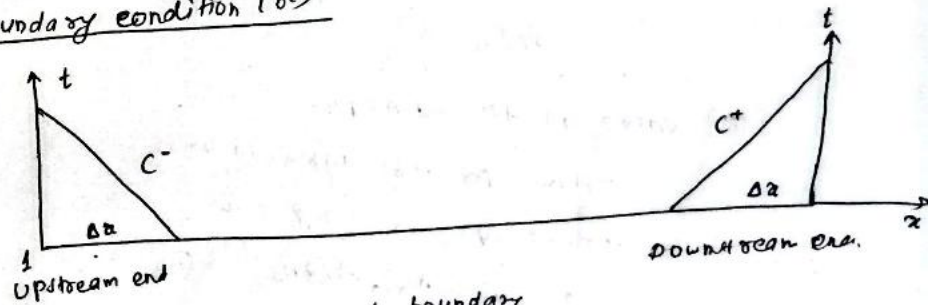
If 2 reach,



A at $t=0$ $\rightarrow$ C-line solve (M etc Δt)



6.3 Boundary condition (BC)



$\rightarrow$ At $t=0$ end, C- line intersects boundary
 $\rightarrow$ At $t=L$ end, C+ " " "

① Reservoir at u/lr end.

HGL = const for short duration.

[$t=0$ const throughout].

$$H_{P1} = H_R + \Delta H \sin \omega t \quad [Q_{P1} \text{ from eq. (1)}]$$

② Discharge as specified function of time at u/lr end.

$$Q_{P1} = Q_0 + \Delta Q \sin \omega t \rightarrow \text{known as BC}$$

[H_{P1} from eq. (1)]

③ Centrifugal pump at u/lr end with head-discharge curve (table or eq.) known

$$H_{P1} = H_S + Q_{P1} (a_1 + a_2 Q_{P1})$$

H_S = shutoff head.

a_1 and a_2 : const. of curve

Solve this eq. with eq. (1) simultaneously at the Boundary

④ Dead end at d/r end of pipe

$$(Q_P)_{nr} = 0$$

⑤ valve at d/r end of pipe.

Datum for HGL taken at valve

orifice eq. for steady flow.

$$Q_0 = C C_D A_v]_0 \sqrt{2gH_0} \quad \text{--- (1)}$$

Q_0 : steady state flow.

H_0 : steady state head loss across the valve.

$(C C_D A_v)_0$ = discharge coeff $\propto$ area of valve opening

For another opening

$$Q_P = C_D A_G \sqrt{2g\Delta H} \quad \text{--- (2)}$$

ΔH = instantaneous drop in HGL across the valve.

$$T = \frac{C_D A_G}{(C_D A_v)_0} \quad \text{--- (iii)}$$

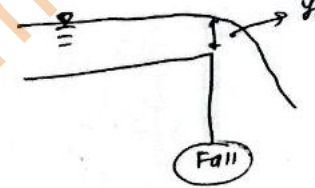
$$Q_P = \frac{Q_0}{\sqrt{H_0}} T \sqrt{\Delta H} \quad \text{--- (iv)}$$

steady flow $T=1$,

closed valve with no flow, $T=0$

Other

say for open channel



fall to just after critical depth.

For sec. ch.

$$y_c = \left(\frac{Q}{g} \right)^{2/3}$$

$$y_c = w \quad \text{(BC)}$$

MOC

1.8

1

$$D = 30 \text{ cm} = 0.3 \text{ m}$$

$$Q_A = 0.7 \text{ m}^3/\text{s}$$

$$Q_B = 0.76 \text{ m}^3/\text{s}$$

$$\phi_c = 0.74 \text{ m } 3/5$$

$$K_A = 20 \text{ m}$$

$$H_D = 20.6 \text{ m}$$

$$H_c = 20.4 \text{ m}$$

$$\Delta x = 1000 \text{ m}$$

$$\Delta t = 1 \text{ sec}$$

$$f = 0.02$$

$$C = 800 \text{ m/s}$$

$$\frac{\Delta s}{\Delta t} = c$$

$$\Rightarrow [\Delta s = 800 \times 1 = 800 \text{ m}]$$

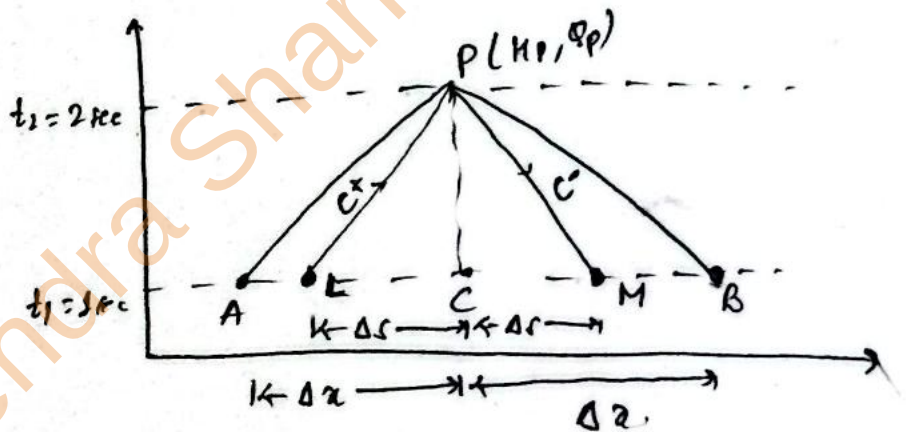
For L point,

$$\frac{P_L - P_C}{\Delta S} = \frac{P_A - P_C}{\Delta x}$$

$$\text{or } \frac{P_L - P_C}{P_A - P_C} = \frac{\Delta S}{\Delta x}$$

$$\therefore \frac{92 - 0.74}{0.7 - 0.74} = \frac{800}{1000}$$

$$\Rightarrow \{Q = 0.708 \text{ m}^3/\text{s}\}$$



$$\frac{H_L - H_C}{H_A - H_C} = \frac{\Delta \sigma}{\Delta \alpha}$$

$$\text{on } \frac{H_L - 20.4}{20 - 20.4} = \frac{800}{1000}$$

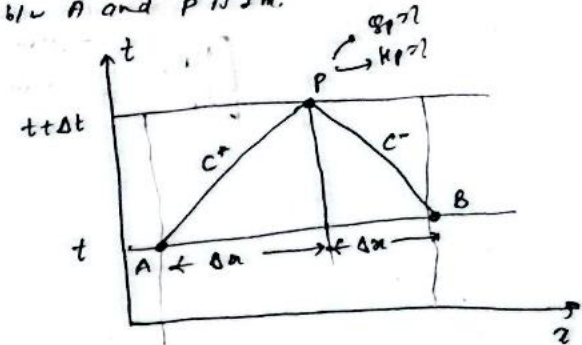
$$\Rightarrow [H_L = 20.0 \text{ rpm}]$$

For MPT

$$Q_H = 0.756 \text{ m}^3/\text{s}$$

$$H_H = 20.56 \text{ m}$$

Q2 Following are data given at two points A and B along a pipe of dia 30cm carrying water as shown in the figure below. The discharge are $0.4 \text{ m}^3/\text{s}$ and $0.45 \text{ m}^3/\text{s}$, head being 26.5 m and 27.5 m respectively. Compute discharge and head at P using finite diff form of characteristic equation. Given: $\Delta x = 500 \text{ m}$, $\Delta t = 0.4 \text{ sec}$, $f = 0.02$, $C = 1200 \text{ m/s}$, elevation difference b/w A and P is 1 m .



→ Sol

$$D = 30 \text{ cm} = 0.3 \text{ m}$$

$$\begin{array}{l} Q_A = 0.4 \text{ m}^3/\text{s} \\ Q_B = 0.45 \text{ m}^3/\text{s} \end{array} \quad \left| \quad \begin{array}{l} H_A = 26.5 \text{ m} \\ H_B = 27.5 \text{ m} \end{array} \right.$$

$$Q_P = ? , H_P = ?$$

$$\Delta x = 500 \text{ m}, \Delta t = 0.4 \text{ sec}, f = 0.02, C = 1200 \text{ m/s}$$

$$\Delta z = 1.0 \text{ m}$$

$$\text{For } C^+: H_P = H_A - B(Q_P - Q_A) - R(Q_A | Q_A) \quad \text{--- (I)}$$

$$\text{For } C^-: H_P = H_B + B(Q_P - Q_B) + R(Q_B | Q_B) \quad \text{--- (II)}$$

$$B = \frac{C}{gA} = \frac{1200}{9.81 \times \pi \times \frac{0.3^2}{4}} = 1722.875$$

$$R = \frac{f \Delta x}{2gDA^2} = \frac{0.02 \times 500}{2 \times 9.81 \times 0.3 \times \left(\frac{\pi \times 0.3^2}{4}\right)^2} = 327.0257$$

From ①

$$H_p = 26.5 - 1722.875(\phi_p - 0.4) - 327.025 \times 0.4 \times 0.4 \quad \text{--- (iii)}$$

$$[H_p + 1722.875\phi_p = 661.72]$$

From ②

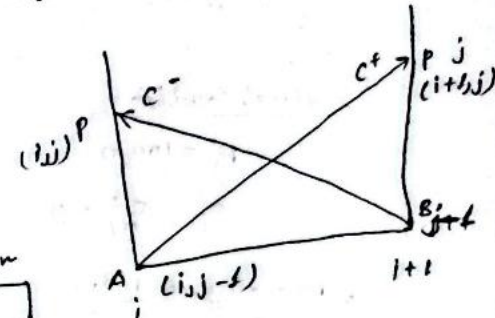
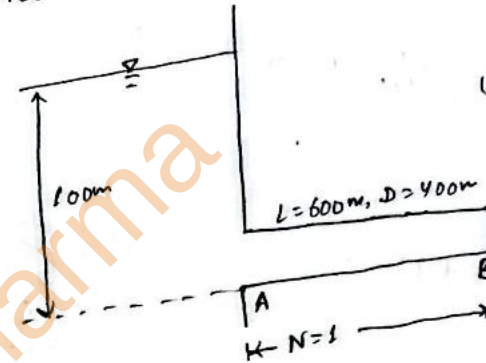
$$H_p = 27.5 + 1722.875(\phi_p - 0.4) + 327.025 \times 0.45 \times 0.45 \quad \text{--- (iv)}$$

$$[H_p - 1722.875\phi_p = -679.54]$$

$$\begin{bmatrix} H_p = -8.91m \\ \phi_p = 0.389m^2/s \end{bmatrix}$$

Water Hammer

Q3 A pipe conveys water from a reservoir as shown in figure. Take $f = 0.02$, $c = 1200 \text{ m/s}$. The HGL at the reservoir is given as $H_{pp} = 100 + 3\sin(\pi t)$. The discharge at downstream is zero at all times. By using only one reach, compute discharge from A and elevation of HGL at B at 3 sec, using discretized equation of MOC.



$$f = 0.02, \quad c = 1200 \text{ m/s}$$

$$H_{pp} = 100 + 3\sin(\pi t)$$

$$N = 1$$

$$\phi_B = 0$$

$$t = 3 \text{ sec}, \quad \rightarrow \phi_A?? \quad \& \quad H_B??$$

$$\Delta x = \frac{L}{N} = \frac{600}{1} = 600 \text{ m}$$

$$\Delta t = \frac{\Delta x}{c} = \frac{600}{1200} = 0.5 \text{ sec}$$

For C^+ :

$$H_p = H_A - B(\phi_p - \phi_A) - R\phi_A / \phi_A$$

$$\text{or } H_{i+1}^j = H_i^{j-1} - B(\phi_{i+1}^j - \phi_i^{j-1}) - R\phi_i^{j-1} / \phi_i^{j-1} \quad \text{--- (i)}$$

For C^- :

$$H_p = H_B + B(\phi_p - \phi_B) + R\phi_B / \phi_B$$

$$\text{or } H_i^j = H_{i+1}^{j-1} + B(\phi_i^j - \phi_{i+1}^{j-1}) + R\phi_{i+1}^{j-1} / \phi_{i+1}^{j-1} \quad \text{--- (ii)}$$

$$B = \frac{c}{gH} = \frac{1200}{9.81 \times \frac{\pi}{4} \times 0.4^2} = 973.42$$

$$z = \frac{fLx}{2gD} = \frac{0.02 \times 600}{2 \times 9.81 \times 0.4 \times \left(\frac{\pi}{4} \times 0.4^2\right)^2} = 96.83$$

Initial condition,

$$H_i^j = 100 \text{ m}, \quad H_{i+1}^j = 100 \text{ m}$$

$$Q_i^j = Q_{i+1}^j = 0$$

Boundary condition

$$H_i^j = 100 + 3 \sin(\pi t)$$

$$Q_{i+1}^j = 0$$

Sq = ① becomes

$$H_{i+1}^j = H_i^{j-1} - 973.42 (Q_{i+1}^{j-1} - Q_i^{j-1})$$

$$- 96.83 \times Q_i^{j-1} | Q_{i+1}^{j-1}$$

$$\text{or, } H_{i+1}^j = H_i^{j-1} + 973.42 Q_i^{j-1} - 96.83 Q_i^{j-1} | Q_{i+1}^{j-1} \quad \text{--- (ii)}$$

Sq = ② becomes

$$H_i^j = H_{i+1}^{j-1} + 973.42 (Q_i^{j-1} - Q_{i+1}^{j-1}) + 96.83 Q_{i+1}^{j-1} | Q_{i+1}^{j-1}$$

$$H_i^j = H_{i+1}^{j-1} + 973.42 Q_i^{j-1}$$

$$\text{or, } Q_i^j = \frac{H_i^j - H_{i+1}^{j-1}}{973.42} \quad \text{--- (iv)}$$

t (sec)	j	$H_i^j = 100 + 3 \sin(\pi t)$	H_{i+1}^j	Q_i^j	Remarks
0	1	100	100	0	Initial cond.
0.5	2	103	100	3.08×10^{-3}	
1	3	100	106	0	
1.5	4	97	100	-3.24×10^{-3}	
2	5	100	88.014	0	
2.5	6	103	100	0.0154	
3	7	100	117.968	0	

Example calc

For j=2,

$$H_i^2 = 100 + 3 \sin(\pi \times 0.5) = 103$$

$$H_{i+1}^2 = H_i^1 + 973.42 Q_i^1 - 96.83 Q_i^1 | Q_{i+1}^1$$

$$= 100 + 973.42 \times 0 - 96.83 \times 0 \times 0$$

$$= 100$$

$$Q_i^2 = \frac{H_i^2 - H_{i+1}^1}{973.42} = \frac{103 - 100}{973.42} = 3.08 \times 10^{-3} \text{ m}^3/\text{s}$$

For j=3

$$H_i^3 = 100 + 3 \sin(\pi \times 1) = 100$$

$$H_{i+1}^3 = H_i^2 + 973.42 \times (3.08 \times 10^{-3}) - 96.83 \times (3.08 \times 10^{-3}) | (3.08 \times 10^{-3})$$

$$= 105.997 \text{ m}$$

$$\approx 106 \text{ m}$$

$$Q_i^3 = \frac{H_i^3 - H_{i+1}^2}{973.42} = \frac{100 - 100}{973.42} = 0$$

Thus

$$Q_B = 0 \text{ m}^3/\text{s}$$

$$H_B = 117.968 \text{ m}$$

Simulation of Groundwater flow

Darcy's law:-

velocity through porous media $\propto$ hydraulic gradient

$$[v = -k \frac{dh}{dx}]$$

↑
hydraulic conductivity

$$[Q = Av = -kA (dh/dx)]$$

In terms of transmissivity (T)

$$[v = -(T/H) * (dh/dx)]$$

$$[as T = k * H]$$

H → thickness of aquifer.

Steady continuity eqⁿ for incompressible flow

$$[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0]$$

Stream line! → tangent to any pt. on it represents direction of velocity vector at that pt.

stream function (ψ) = const.

→ steady flow: path line = streamline
→ drawn in dir of flow

Equipotential line

→ velocity potential (hydraulic head) constant
→ $\perp$ to streamlines

Total energy head or flow potential

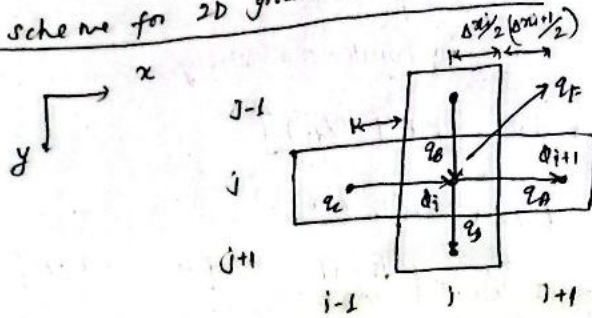
$$[P/\gamma + z = h = \text{piezometric head}]$$

→ contour of h represent ϕ

Thus

$$V = -K \left(\frac{d\phi}{dy} \right)$$

7-2 FD scheme for 2D groundwater simulation



Continuity

Inflow - Outflow = rate of change of storage

$$[\Delta x_i \Delta y_j (q_e - q_0 - q_n + q_s - q_f) = S \frac{\partial \phi}{\partial t} \Delta x_i \Delta y_j] \quad \text{--- (1)}$$

S = storage coeff

= $\frac{\text{Volume of water released or taken}}{\text{Area} \times \text{change in hwt}}$

Rate of change of storage = $S \times \text{rate of change of head} \times \text{area}$

Darcy's eqⁿ

↳ In terms of ϕ : $Q = -K A \left(\frac{\partial \phi}{\partial x} \right)$

↳ In terms of T : $Q = -K \Delta x_i \Delta y_j \left(\frac{\partial \phi}{\partial x} \right)$

Non-homogenous & anisotropic aquifer,

$$(T_x = K_x \Delta x_i)$$

T_x = weighed avg. of transmissivity for grid i and $i+1$

$$\rightarrow Q_n = -T_x (\Delta y_j) \frac{\phi_{i+1,j} - \phi_{i,j}}{\frac{\Delta x_{i+1}}{2} + \frac{\Delta x_i}{2}}$$

$$\text{or } Q_n = -2T_x (\Delta y_j) \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x_{i+1} + \Delta x_i}$$

Also, $\phi_{i+1,j} - \phi_{i,j} = \Delta \phi$ = total head loss in Δx

$$\Delta \phi = \frac{Q_n}{2T_x (\Delta y_j)} (\Delta x_{i+1} + \Delta x_i)$$

$$= \frac{Q_n}{2T_{x_{ij}} (\Delta y_j)} (\Delta x_i) - \frac{Q_n}{2T_{x_{i+1,j}} (\Delta y_j)} (\Delta x_{i+1})$$

Solving for T_x ,

$$T_x = \frac{T_{x_{ij}} T_{x_{i+1,j}}}{\Delta x_i T_{x_{i+1,j}} + \Delta x_{i+1} T_{x_{ij}}} (\Delta x_i + \Delta x_{i+1})$$

Thus

$$q_n = \frac{Q_n}{\Delta x_i \Delta y_j} = -A_{ij} (\phi_{i+1,j} - \phi_{i,j})$$

$$\text{where, } A_{ij} = \frac{2T_{x_{ij}} T_{x_{i+1,j}}}{\Delta x_i (\Delta x_i T_{x_{i+1,j}} + \Delta x_{i+1} T_{x_{ij}})}$$

Similarly written for q_e, q_s, q_w

Eqn ①,

$$C_{i,j}(\phi_{i,j} - \phi_{i,j-1}) + D_{i,j}(\phi_{i,j+1} - \phi_{i,j}) + A_{i,j}(\phi_{i+1,j} - \phi_{i,j}) + B_{i,j}(\phi_{i,j-1} - \phi_{i,j}) - q_F = \frac{S \partial \phi}{\partial x}$$

or, $[A\phi_{i+1,j} + B\phi_{i,j-1} + C\phi_{i-1,j} + D\phi_{i,j+1} - E\phi_{i,j} = \frac{S \partial \phi}{\partial x} + q_F]$ (vi)

where, $[E = A + B + C + D]$

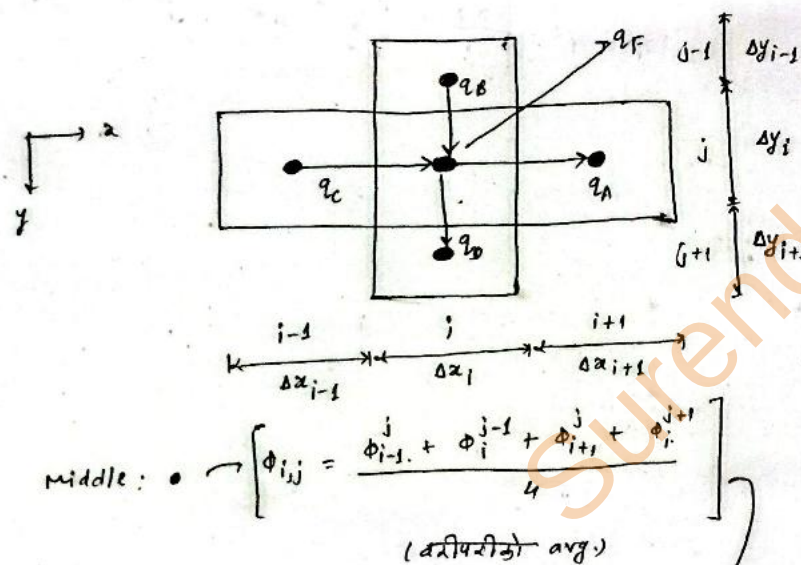
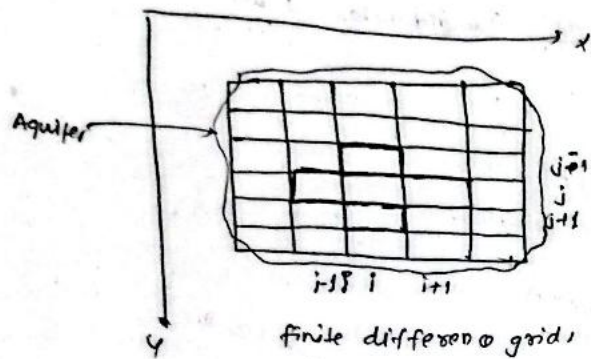
Homogenous $\Rightarrow \Delta x = \Delta y$
 Isotropic $\Rightarrow T_x = T_y = T$

Thus, $A = B = C = D = \frac{T}{\Delta x^2}$ & $E = A + B + C + D = \frac{4T}{\Delta x^2}$

Steady flow $\Rightarrow \frac{\partial \phi}{\partial x} = 0$

No lateral inflow, $q_F = 0$

7.2 Finite difference scheme for 2D GW simulation



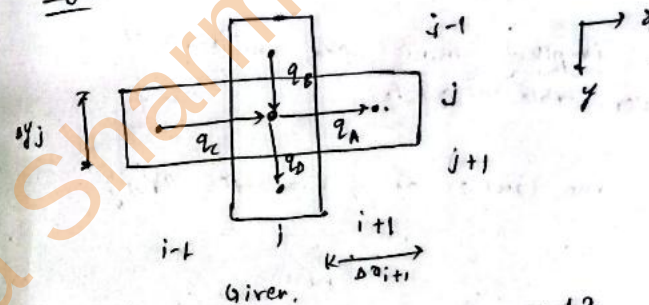
Homogeneous, isotropic & steady flow.

$q \rightarrow$ discharge per unit area.
 $\hookrightarrow$ Darcy's flux.

$$\begin{cases} q_c = -C_{i,j} (\phi_{i,j} - \phi_{i-1,j}) \\ q_b = -B_{i,j} (\phi_{i,j} - \phi_{i,j-1}) \\ q_a = \dots \\ q_d = \dots \end{cases}$$

(कॉन्वेंशन कॉन्वेंशन)

eg:



Given:

$$\begin{aligned} \phi_{i-1,j} &= 12, \quad \phi_{i,j} = 13 \\ \phi_{i+1,j} &= 14, \quad \phi_{i,j-1} = 13.5 \\ \phi_{i,j+1} &= 14 \\ T_a &= 0.013 \text{ m}^3/\text{s}, \quad T_y = 0.015 \text{ m}^3/\text{s} \\ \Delta x &= 20 \text{ m}, \quad \Delta y = 20 \text{ m} \end{aligned}$$

calculate Darcy's flux q_a, q_b, q_c & q_d

$$q_a = -A_{i,j} (\phi_{i+1,j} - \phi_{i,j})$$

For, Δx constant & $T_a = \text{constant}$ for one dir.

$$\begin{aligned} A_{i,j} &= \frac{T_a}{(\Delta x)^2} = \frac{0.013}{20^2} = 3.25 \times 10^{-5} \text{ per sec.} \\ q_a &= -3.25 \times 10^{-5} (14 - 13) = -3.25 \times 10^{-5} \text{ m}^3/\text{s}/\text{m}^2 \end{aligned}$$

$$q_B = -B_{ij} (\phi_{ij} - \phi_{ij-1})$$

$$B_{ij} = \frac{Ty}{\Delta y^2} = \frac{0.015}{(25)^2} = 2.4 \times 10^{-5} \text{ m/sec}$$

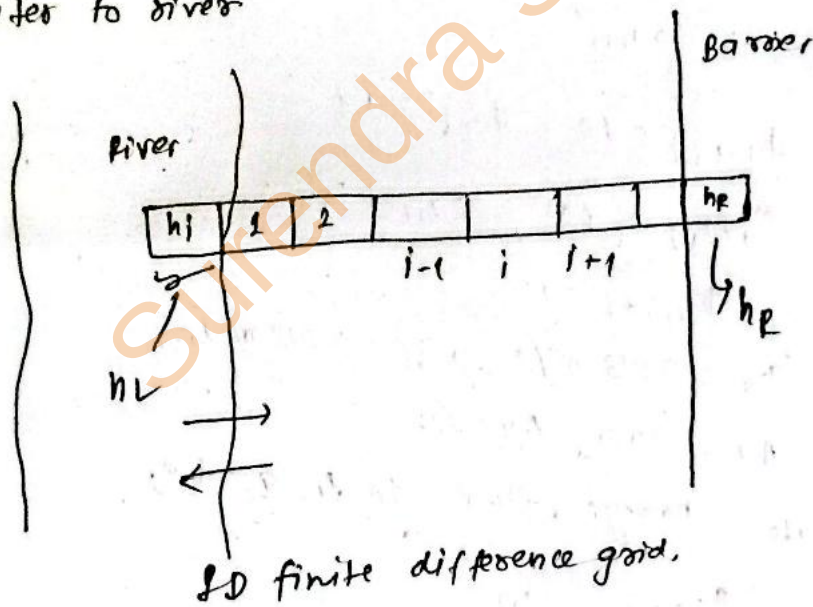
$$q_B = -2.4 \times 10^{-5} (13 - 13.5) = 1.2 \times 10^{-5} \text{ m/sec}$$

(flow higher potential to lower potential)

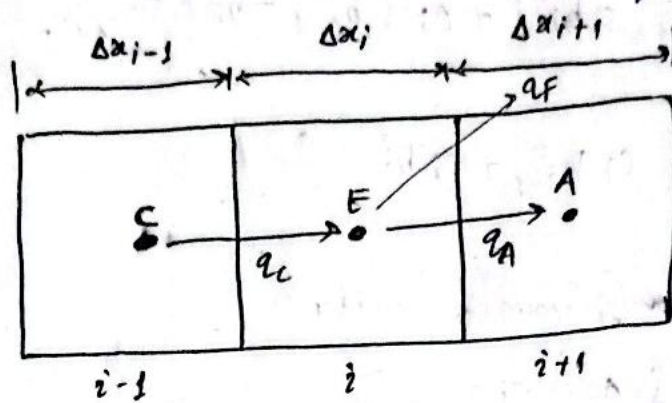
$\therefore$ -ve in q formula.

7.4 [One-Dimensional implicit model for simulating river stage-water table interactions]

river recharging GW aquifer or GW aquifer supplying water to river



contd



Partial differential equation for 1D unsteady groundwater flow is

$$\frac{\partial}{\partial x} \left(T \frac{\partial \phi}{\partial x} \right) = S \frac{\partial \phi}{\partial t} + q_f$$

where, T = Transmissibility, ϕ = water table elev. for an unconfined aquifer or the elev. of piezometric surface for confined aquifer.

S = storage coefficient

q_f = net groundwater withdrawal per unit area.

[1D $\rightarrow$ $\frac{\partial}{\partial x}$ P3(1) grid (direction) $\phi(x, t)$]

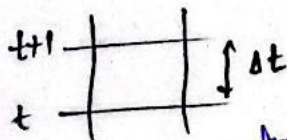
Continuity eqⁿ:

inflow - outflow = rate of change of storage.

$$[q_c - q_A - q_f = S \frac{\partial \phi}{\partial t}]$$

$$-C_i (\phi_i - \phi_{i-1}) - A_i (\phi_{i+1} - \phi_i) = S \frac{\partial \phi}{\partial t} + q_f$$

$$= \frac{S_i (\phi_i^+ - \phi_i)}{\Delta t} + q_f$$



$h \rightarrow$ beginning of Δt
 $h^+ \rightarrow$ end of Δt
 water table elev.

Implicit to +

$$A_i (h_{i+1}^+ - h_i^+) + C_i (h_{i-1}^+ - h_i^+) = \frac{S_i}{\Delta t} (h_i^+ - h_i) + q_i$$

$$\rightarrow [A_i h_{i+1}^+ + C_i h_{i-1}^+ + E_i h_i^+ = \Phi_i]$$

For homogeneous & isotropic

$$\begin{cases} A_i = C_i = \frac{T}{\Delta x^2} \\ E_i = -\left(\frac{2T}{\Delta x^2} + \frac{S_i}{\Delta t}\right) \end{cases}$$

Implicit scheme $\rightarrow$ unknown $\rightarrow$ many
 $\rightarrow$ complex problem

$\rightarrow$ Continuity eqⁿ become

$$\rightarrow [C_i h_{i-1}^+ + E_i h_i^+ + A_i h_{i+1}^+ = \Phi_i]$$

Writing this eqⁿ for each grid section in tri-diagonal
coefficient matrix.

For $i=1$

$$\textcircled{X} C_1 h_L + E_1 h_1^+ + A_1 h_2^+ = \Phi_1$$

For $i=2$

$$C_2 h_1 + E_2 h_2^+ + A_2 h_3^+ = \Phi_2$$

For $i=n$

$$C_n h_{n-1} + E_n h_n^+ + A_n h_{n+1}^+ = \Phi_n$$

In matrix form

$$\begin{bmatrix} E_1 & A_1 & 0 & 0 & 0 & 0 & 0 \\ C_2 & E_2 & A_2 & 0 & 0 & 0 & 0 \\ 0 & C_3 & E_3 & A_3 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & C_n & A_n \end{bmatrix} \begin{bmatrix} h_1^+ \\ h_2^+ \\ h_3^+ \\ \vdots \\ h_n^+ \end{bmatrix} = \begin{bmatrix} \Phi_1 - C_1 h_L \\ \Phi_2 \\ \Phi_3 \\ \vdots \\ \Phi_n - A_n h_{n+1}^+ \end{bmatrix}$$

h_L = river stage at potential boundary

A_n right boundary is barrier boundary $A_n = 0$.

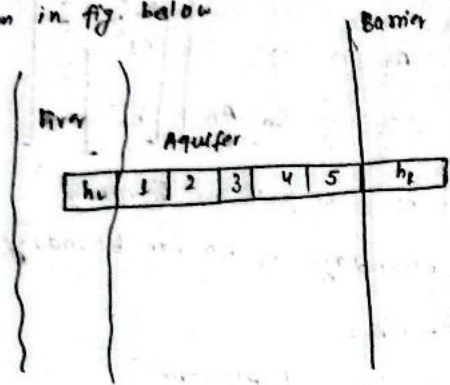
Find h

Seepage rate b/w grid elements

$$q_i = C_i \Delta x_i (h_{i-1}^+ - h_i^+)$$

Loss or gain at the river = seepage at grid element 1. ($i=1$)

Q. A schematic for simulating river stage-water table fluctuation is shown in fig. below



The following data are given for simulation:
 homogeneous isotropic aquifer river stage (h_L) = 2.03m,
 Length of aquifer = 500m, $\Delta x = 100$ m, $\Delta t = 1$ day,
 transmissivity of aquifer = 0.03m/s,
 storage coeff = 0.01

Initial value of water table at five grids:
 99.1, 99.11, 99.2, 99.3, 99.4

Compute coeff. of the 1D implicit finite difference model and display the matrix of known & unknowns.

→ Soln

The finite diff. eqn:

$$C_i h_{i-1}^t + E_i h_i^t + A_i h_{i+1}^t = \Phi_i$$

For homogeneous and isotropic aquifer, coeff are computed as,

$$A_i = C_i = \frac{T}{\Delta x^2} = \frac{0.03}{100^2} = 3 \times 10^{-6} \text{ (same for all grids)}$$

$$E_i = - \left(\frac{2T}{\Delta x^2} + \frac{S_i}{\Delta t} \right) = - \left(\frac{2 \times 0.03}{100^2} + \frac{0.01}{1 \times 24 \times 3600} \right)$$

$$= -7 \times 10^{-6} \text{ (same for all grids)}$$

$$\Phi_i = - \frac{S_i}{\Delta t} h_i + \Phi_F$$

Take $\Phi_F = 0$

$$\Phi_i = - \frac{S_i}{\Delta t} h_i, \text{ where } h_i = \text{initial values.}$$

$$\Phi_1 = - \frac{0.01}{1 \times 24 \times 3600} \times 99.1 = -1.14 \times 10^{-5}$$

$$\Phi_2 = - \frac{0.01}{1 \times 24 \times 3600} \times 99.11 = -1.147 \times 10^{-5}$$

$$\Phi_3 = - \frac{0.01}{1 \times 24 \times 3600} \times 99.2 = -1.148 \times 10^{-5}$$

$$\Phi_4 = - \frac{0.01}{1 \times 24 \times 3600} \times 99.3 = -1.149 \times 10^{-5}$$

$$\Phi_5 = - \frac{0.01}{1 \times 24 \times 3600} \times 99.4 = -1.15 \times 10^{-5}$$

Boundary conditions

$$h_L = h_0 = 2.03 \text{m}$$

$$\text{For } i=1: C_1 h_0^t + E_1 h_1^t + A_1 h_2^t = \Phi_1$$

$$E_1 h_1^t + A_1 h_2^t = \Phi_1 - C_1 h_L$$

$$\text{For } i=2: C_2 h_1^t + E_2 h_2^t + A_2 h_3^t = \Phi_2$$

$$\text{For } i=3: C_3 h_2^t + E_3 h_3^t + A_3 h_4^t = \Phi_3$$

$$\text{For } i=4: C_4 h_3^t + E_4 h_4^t + A_4 h_5^t = \Phi_4$$

$$\text{For } i=5: C_5 h_4^t + E_5 h_5^t + A_5 h_6^t = \Phi_5$$

$$C_5 h_4^t + E_5 h_5^t + 0 = \Phi_5$$

Coefficient matrix is 5x5 matrix (tridiagonal elements non-zero, others zero)

$$\begin{bmatrix} E_1 & A_1 & 0 & 0 & 0 \\ -C_2 & E_2 & A_2 & 0 & 0 \\ 0 & C_3 & E_3 & A_3 & 0 \\ 0 & 0 & C_4 & E_4 & A_4 \\ 0 & 0 & 0 & C_5 & E_5 \end{bmatrix} \begin{bmatrix} h_1^t \\ h_2^t \\ h_3^t \\ h_4^t \\ h_5^t \end{bmatrix} = \begin{bmatrix} \Phi_1 - C_1 h_L \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \end{bmatrix}$$

$$\begin{bmatrix} -7 \times 10^{-6} & 5 \times 10^{-6} & 0 & 0 & 0 \\ 5 \times 10^{-6} & -7 \times 10^{-6} & 5 \times 10^{-6} & 0 & 0 \\ 0 & 5 \times 10^{-6} & -7 \times 10^{-6} & 3 \times 10^{-6} & 0 \\ 0 & 0 & 3 \times 10^{-6} & -7 \times 10^{-6} & 3 \times 10^{-6} \\ 0 & 0 & 0 & 3 \times 10^{-6} & -7 \times 10^{-6} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{bmatrix} = \begin{bmatrix} 5.98 \times 10^{-4} \\ 1.147 \times 10^{-5} \\ 1.148 \times 10^{-5} \\ 1.149 \times 10^{-5} \\ 1.15 \times 10^{-5} \end{bmatrix}$$

~ एक दिन पछि ग्रिड में water table की स्तर

differential eq → difference eq (discrete)

From 7.2

Use continuity eq:

$$\text{Inflow} - \text{outflow} = \text{rate of change of storage}$$

q_F → outflow
SS बाह्य निकले

For middle grid.

$$\Delta x_i \Delta y_i [q_c - q_D + q_B - q_F - q_A] = S \frac{\partial \phi}{\partial t} \Delta x_i \Delta y_i \quad \text{--- (1)}$$

S → storage coeff.

ϕ → potential head.

$$S = \frac{\text{vol}^n \text{ of water released or taken}}{\text{area} \times \text{change in hwt}}$$

q → flow rate / area (Darcy flux)

change in storage = $S \times \text{rate of change of head} \times \text{area}$

Darcy's eq in form of ϕ (which is potential head h).

$$Q = -KA \frac{\partial \phi}{\partial x}$$

$$Q = 1000 A = \frac{\partial h}{\partial x} \rightarrow q$$

In terms of transmissivity,

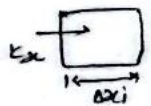
$$Q = -K \Delta x_i \Delta y_i \frac{\partial \phi}{\partial x} \quad \text{--- (1)}$$

In x-dir:-

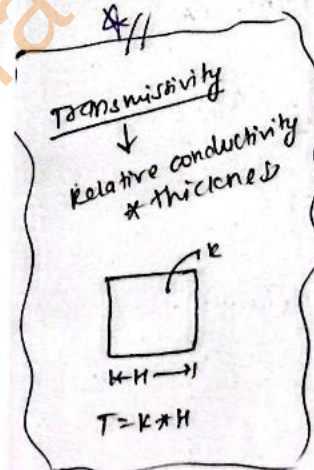
$$T_x = K_x \Delta x_i$$

T_x → weighted avg. transmissivity for grid 1 and 1

consider (i,j) and (i+1,j) grid



$$T_y = K_y \Delta y_j$$



Finite difference form of Darcy's eqⁿ,

$$q_p = -T_x (\Delta y_j) \frac{\phi_{i+1,j} - \phi_{i,j}}{\frac{\Delta x_{i+1}}{2} + \frac{\Delta x_i}{2}} = -2 \Delta T_x \Delta y_j \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x_{i+1} + \Delta x_i} \quad \text{--- (iii)}$$

$\phi_{i+1,j} - \phi_{i,j} = \Delta \phi$ = total head loss in x dir.

Thus, eqⁿ (iii) becomes

$$\Delta \phi = - \frac{q_p}{2 T_x \Delta y_j} (\Delta x_{i+1} + \Delta x_i)$$

$$\Delta \phi = - \frac{q_p}{2 T_{x_{i,j}} (\Delta y_j)} (\Delta x_i) - \frac{q_p}{2 T_{x_{i+1,j}} (\Delta y_j)} (\Delta x_{i+1})$$

Solving for T_x ,

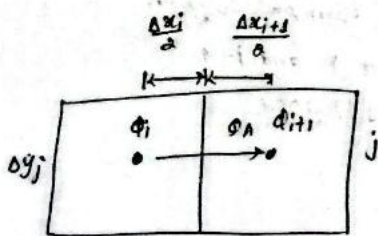
$$T_x = \frac{T_{x_{i,j}} + T_{x_{i+1,j}}}{\Delta x_i T_{x_{i+1,j}} + \Delta x_{i+1} T_{x_{i,j}}} (\Delta x_i + \Delta x_{i+1})$$

Solving T_x in eqⁿ (iii) and finding q_A

$$\text{Darcy's flux; } q_A = \frac{q_p}{\Delta x_i \Delta y_j} = -A_{i,j} (\phi_{i+1,j} - \phi_{i,j})$$

where,

$$A_{i,j} = \frac{2 T_{x_{i,j}} * T_{x_{i+1,j}}}{\Delta x_i (\Delta x_i T_{x_{i+1,j}} + \Delta x_{i+1} T_{x_{i,j}})}$$



Similarly,

$$q_c = -C_{i,j} (\phi_{i,j} - \phi_{i-1,j})$$

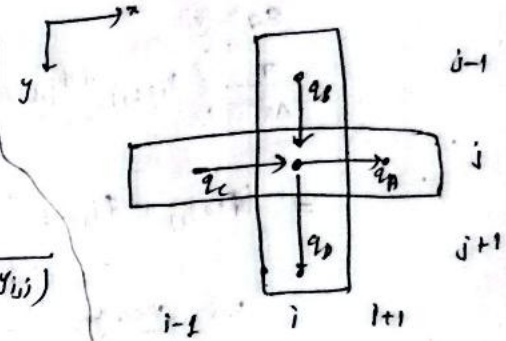
$$q_D = -D_{i,j} (\phi_{i,j+1} - \phi_{i,j})$$

$$q_B = -B_{i,j} (\phi_{i,j} - \phi_{i,j-1})$$

$$B_{i,j} = \frac{2 T_{y_{i,j}} T_{y_{i,j-1}}}{\Delta y_j (\Delta y_j T_{y_{i,j-1}} + \Delta y_{j-1} T_{y_{i,j}})}$$

$$C_{i,j} = \frac{2 T_{x_{i,j}} T_{x_{i-1,j}}}{\Delta x_i (\Delta x_i T_{x_{i-1,j}} + \Delta x_{i-1} T_{x_{i,j}})}$$

$$D_{i,j} = \frac{2 T_{y_{i,j}} T_{y_{i,j+1}}}{\Delta y_j (\Delta y_j T_{y_{i,j+1}} + \Delta y_{j+1} T_{y_{i,j}})}$$



q_F → Net withdrawal
NOT from grid block

Substituting values in eqⁿ (i),

$$C_{i,j} (\phi_{i-1,j} - \phi_{i,j}) + D_{i,j} (\phi_{i,j+1} - \phi_{i,j}) + A_{i,j} (\phi_{i+1,j} - \phi_{i,j}) + B_{i,j} (\phi_{i,j-1} - \phi_{i,j}) - q_F = \frac{S \partial \phi}{\partial t} \quad \text{--- (v)}$$

Dropping subscripts of coeff,

$$A \phi_{i+1,j} + B \phi_{i,j-1} + C \phi_{i-1,j} + D \phi_{i,j+1} - E \phi_{i,j} = \frac{S \partial \phi}{\partial t} + q_F \quad \text{--- (vi)}$$

where, $E = A + B + C + D$

Simplification of homogenous and isotropic aquifer

Homogenous aquifer → $\Delta x = \Delta y$
Isotropic " → $T_x = T_y = T$

$$\text{Thus, coeff. } (A, B, C, D) = \frac{\Delta T}{\Delta x^2} = \frac{\Delta T}{\Delta y^2}$$

$$E = A + B + C + D = \frac{4T}{\Delta x^2}$$

Eqn (vi)

$$\frac{T}{\Delta x^2} (\phi_{i+1,j} + \phi_{i,j-1} + \phi_{i-1,j} + \phi_{i,j+1} - 4\phi_{i,j}) = \left(S \frac{\partial \phi}{\partial t} + q_F \right)$$

$$\Rightarrow (\phi_{i+1,j} + \phi_{i,j-1} + \phi_{i-1,j} + \phi_{i,j+1} - 4\phi_{i,j}) = \frac{\Delta x^2}{T} \left(S \frac{\partial \phi}{\partial t} + q_F \right)$$

For steady flow with $q_F = 0$, $\left(S \frac{\partial \phi}{\partial t} = 0 \right)$ no local entry withdrawal

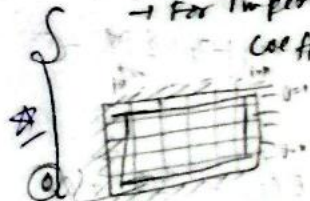
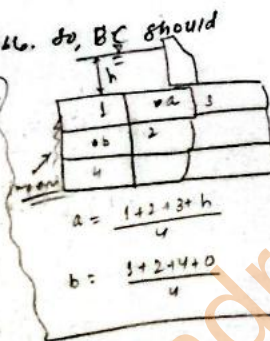
$$[\phi_{i+1,j} + \phi_{i,j-1} + \phi_{i-1,j} + \phi_{i,j+1} - 4\phi_{i,j} = 0] \quad \text{--- (vii)}$$

$\phi_{i,j}$ is equal to avg. of four neighbouring grid values.

→ At boundaries, four grids will not be available. So, BC should be taken to compute coefficients

→ For impervious boundary on all sides, coefficient from that side is zero.

$j=1, B=0$, and $j=n, D=0$,
 $i=1, C=0$ and $i=n, A=0$



→ For grid elements adjacent to a river, only one value of space increment is needed. eg. $A = \frac{2T}{\Delta x^2}$

→ After computing ϕ for diff grids, eqn lines are drawn by interpolation and streamlines drawn by these lines

S.1 The final value of pot. head at four grids:
 $\phi_{i+1,j} = 1.1, \phi_{i,j-1} = 1.09, \phi_{i,j+1} = 1.15, \phi_{i,j} = 1.07$
 The eqn is homogeneous and is simple. Compute $\phi_{i,j}$ considering steady flow & no withdrawal from grid.
 → Use eqn (vii).

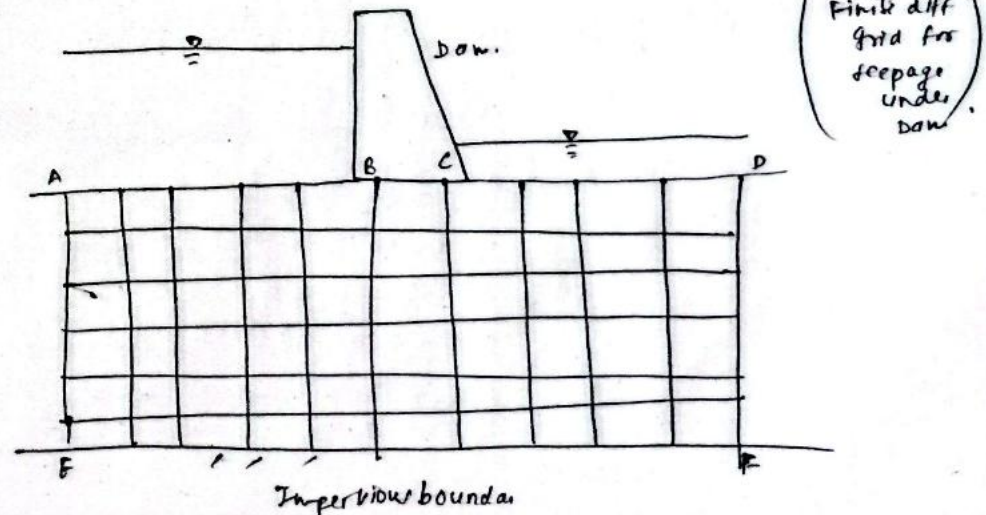
$$\phi_{i,j} = \frac{1.1 + 1.09 + 1.15 + 1.07}{4} = 1.1025$$

7.3 Simulation of seepage under a dam based on steady state 2D model

The eqn for steady-state 2D GWF flow is

$$\frac{\partial}{\partial x} \left(T_x \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(T_y \frac{\partial \phi}{\partial y} \right) = 0$$

T_x, T_y → transmissivity in x and y dir
 ϕ → Pot. head.



Eq. (1)

continuity eq. for interior grid, $-E\phi_{i,j} = \frac{S_0}{\Delta t} + q_r$

$$A\phi_{i+1,j} + B\phi_{i,j-1} + C\phi_{i-1,j} + D\phi_{i,j+1}$$

For steady state with no with draw,

$$A\phi_{i+1,j} + B\phi_{i,j-1} + C\phi_{i-1,j} + D\phi_{i,j+1} - E\phi_{i,j} = 0$$

$$\phi_{i,j}^+ = \frac{(A\phi_{i+1,j} + B\phi_{i,j-1} + C\phi_{i-1,j} + D\phi_{i,j+1})}{E}$$

$\left\{ \begin{array}{l} + : \rightarrow \text{present iteration} \\ \text{without } + : \rightarrow \text{previous iteration.} \end{array} \right.$

Homogeneous Dirichlet

$$\phi_{i,j}^+ = \frac{(\phi_{i+1,j} + \dots)}{4}$$

Iterative procedure to compute potential at each grid and seepage rate:

1. Consider BC for evaluating continuity eq. at boundary.

In the fig, BC and EF are impervious boundaries, AB and CD are equipotential head (const. head).

AE and DF $\rightarrow$ barrier boundary if the grids extend far enough so that no flow occurs across AE and DF

2. Assign initial value of ϕ at each grid. (usually 0 or estimated value)

3. Compute $\phi_{i,j}^+$

4. Compute $e = |\phi_{i,j}^+ - \phi_{i,j}|$

5. If e is less than some specified value, stop the iteration.

6. Else Repeat steps 3 to 4 until condition is satisfied.

7. Compute seepage rate by using Darcy eq.

HZ seepage from grid $i-1$ is

$$q_{i,j}^H = C_{i,j} \Delta x_i \Delta y_j (\phi_{i-1,j} - \phi_{i,j})$$

vt seepage from grid $j-1$ is,

$$q_{i,j}^V = B_{i,j} \Delta x_i \Delta y_j (\phi_{i,j-1} - \phi_{i,j})$$

Simulation of Ground water Flow

NI: Seepage Analysis.

The figure shows 2-D grid for simulating seepage under a dam. The value of potential function (ϕ) are shown in grid. Compute v_x and v_y seepage to grid A. take transmissivity in x-dir = 2900 m²/day for all grids, transmissivity in y-dir = 2400 m²/day in all grid.

$$\Delta x = 100\text{m} \text{ and } \Delta y = 75\text{m}.$$

0	0	0	0
0	2	2	0
0	2.1	2.1	0
0	2.11	2.09 (A)	0
0	2.11	2.12	0
0	0	0	0

Impermeable barrier

→ Soln:-

$$T_{xx} = 2900 \text{ m}^2/\text{day}$$

$$T_{yy} = 2400 \text{ m}^2/\text{day}$$

$$\Delta x = 100\text{m}$$

$$\Delta y = 75\text{m}$$

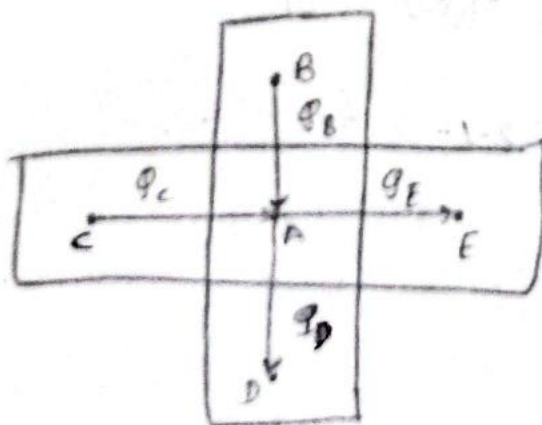
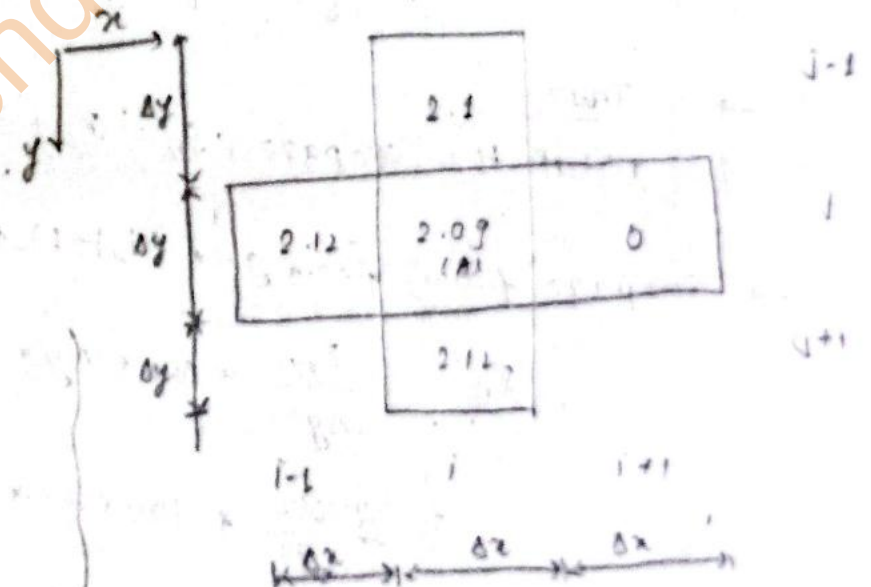


Fig: Direction of seepage flow in grid A)



Enlarged view of grid 'A' with surrounded grid

→ Seepage flow from grid $(i-1, j)$ to grid '0'.

$$Q_c = \frac{T_{xx}}{\Delta x^2} \times \Delta x \times \Delta y \times (\phi_{i-1,j} - \phi_{i,j})$$

$$C_{ij} = \frac{2 T_{x, i-1,j} + T_{x, i,j}}{\Delta x_i (\Delta x_i T_{x, i-1,j} + \Delta x_{i-1} T_{x, i,j})}$$

For homogeneous, $\Delta x_i = \Delta x_{i-1}$

For isotropic, $T_{x, i,j} = T_{x, i-1,j}$

$$Q_c = \frac{2400}{100^2} \times 100 \times 75 \times (2.12 - 2.09)$$

→ Q_c in cm^3/day

$$[Q_c = 62.25 \text{ m}^3/\text{day}]$$

→ As given, There is presence of impermeable barrier in right side of grid A i.e. $\phi = 0$
 $\therefore Q_E = 0$

→ Thus,
 Total H₂ seepage, $Q_H = Q_c + Q_E = 62.25 \text{ m}^3/\text{day}$

→ Seepage flow from grid $(i, j-1)$ to grid A

$$\therefore Q_B = \frac{T_{yy}}{\Delta y^2} \times \Delta x \times \Delta y \times (\phi_{i,j-1} - \phi_{i,j})$$

$$= \frac{2400}{75^2} \times 100 \times 75 \times (2.1 - 2.09)$$

$$= 32 \text{ m}^3/\text{day}$$

→ Seepage flow from grid A to $(i, j+1)$ grid

$$Q_D = \frac{T_{yy}}{\Delta y^2} \times \Delta x \times \Delta y \times (\phi_{i,j} - \phi_{i,j+1})$$

$$= \frac{2400}{75^2} \times 100 \times 75 \times (2.09 - 2.12)$$

$$[Q_D = -36 \text{ m}^3/\text{day}]$$

↳ flow from $(i, j+1)$ to grid A.

Total v_t seepage, $Q_v = Q_B + Q_D$
 $= 32 + 36$
 $= 68 \text{ m}^3/\text{day}$



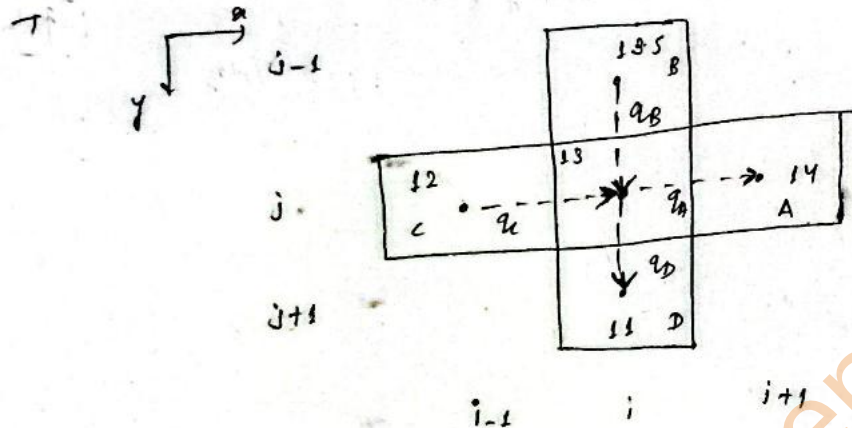
Q. The data below shows a central grid surrounded by four grids for simulating two-dimensional groundwater flow under steady state condition. values of potential function (ϕ) are given below:

$$\phi_{i-1,j} = 12, \phi_{i,j} = 14, \phi_{i,j} = 13, \phi_{i,j-1} = 13.5, \phi_{i,j+1} = 11.$$

Transmissivity in x-dir = $0.013 \text{ m}^2/\text{sec}$ for all grids.
 " " " " y-dir = $0.015 \text{ m}^2/\text{sec}$ " " "

Taking $\Delta x = 20 \text{ m}$, & $\Delta y = 25 \text{ m}$, compute Darcy's flow

q_A, q_B, q_C and q_D from finite difference equation in terms of ϕ .



$$T_{xx} = 0.013 \text{ m}^2/\text{sec}$$

$$T_{yy} = 0.015 \text{ m}^2/\text{sec}$$

$$\Delta x = 20 \text{ m}, \Delta y = 25 \text{ m}$$

$$q_A, q_B, q_C, q_D = ?$$

$$q_C = \frac{T_{xx}}{\Delta x^2} \times (\phi_{i-1,j} - \phi_{i,j})$$

$$= \frac{0.013}{20^2} \times (12 - 13)$$

$$= -0.01625 \text{ m/sec} \quad \leftarrow \text{flow from } (i,j) \text{ to } (i-1,j)$$

$$q_B = \frac{T_{yy}}{\Delta y^2} \times (\phi_{i,j-1} - \phi_{i,j})$$

$$= \frac{0.015}{25^2} \times (13.5 - 13)$$

$$= 1.2 \times 10^{-5} \text{ m/sec} \quad \leftarrow \text{assumed flow dir is correct}$$

$$q_A = \frac{T_{xx}}{\Delta x^2} \times (\phi_{i+1,j} - \phi_{i,j})$$

$$= \frac{0.013}{20^2} \times (14 - 13)$$

$$= 3.25 \times 10^{-5} \text{ m/sec} \quad \leftarrow \text{flow from } (i+1,j) \text{ to } (i,j)$$

$$q_D = \frac{T_{yy}}{\Delta y^2} \times (\phi_{i,j} - \phi_{i,j+1})$$

$$= \frac{0.015}{25^2} \times (13 - 11)$$

$$[q_D = 4.8 \times 10^{-5} \text{ m/sec}] \quad \leftarrow \text{or}$$

RKR
CH-V

Q A river which can be generalized as a trapezoidal channel is 350m wide with side slope 6:1 has a bed slope 1.5% and Manning's $n = 0.0359$. The initial discharge through the river is 420 cumec. Due to a ~~at~~ flood observed flood at upstream, the value of discharge rises to 580 cumec. Calculate the discharge that will occur at 4.65km downstream. Take $\Delta x = 4650$ m and $\Delta t = 1.5$ hour. Use Non-linear kinematic wave solution

→ Soln:

$$b = 350 \text{ m}$$

$$s = 1.5\% = 0.015$$

$$Z = 6$$

$$n = 0.0359$$

$$Q_i = 420 \text{ m}^3/\text{s}$$

Manning's eqⁿ:

$$Q = \frac{1}{n} A R^{2/3} S^{1/2}$$

$$\text{On } 420 = \frac{1}{0.0359} \times \frac{(350y + 6y^2)^{5/3}}{(350 + 2y\sqrt{1+6^2})^{2/3}} \times (0.015)^{1/2}$$

$$\Rightarrow [y = 0.533 \text{ m}]$$

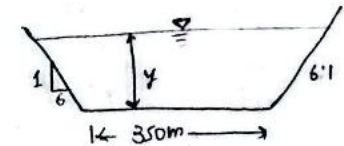
$$P = 350 + 2 \times 0.533 \sqrt{1+6^2} = 356.48 \text{ m}$$

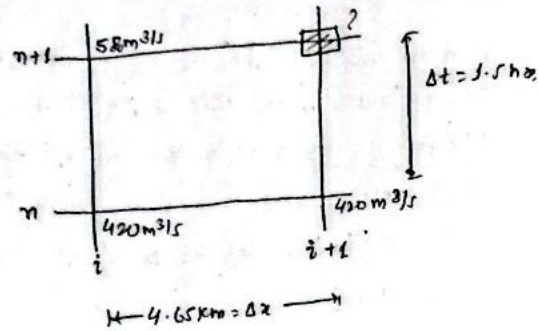
$$A = (350 \times 0.533 + 6 \times 0.533^2) = 188.25 \text{ m}^2$$

$$R = A/P = 0.528$$

$$\alpha = \left(\frac{n P^{2/3}}{\sqrt{s}} \right)^{3/5} = \left(\frac{0.0359 \times 356.48^{2/3}}{\sqrt{0.015}} \right)^{3/5} \approx 5$$

$$\beta = 3/5 = 0.6$$





Continuity eq.

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

$$\text{Or, } \frac{Q_{i+1}^{n+1} - Q_{i+1}^n}{\Delta x} + \frac{A_{i+1}^{n+1} - A_{i+1}^n}{\Delta t} = 0$$

For KWM, $A = \alpha Q^\beta$

$$\text{Or, } \frac{Q_{i+1}^{n+1} - Q_{i+1}^n}{\Delta x} + \frac{\alpha [(Q_{i+1}^{n+1})^\beta - (Q_{i+1}^n)^\beta]}{\Delta t} = 0$$

$$\text{Or, } \frac{Q_{i+1}^{n+1} - 880}{4650} + \frac{5 [(Q_{i+1}^{n+1})^\beta - (420)^{0.6}]}{1.5 \times 60 \times 60} = 0$$

$$A \quad Q_{i+1}^{n+1} =$$

$$861.01 \text{ m}^3/\text{s}$$

$$\text{Or, } \frac{\Delta t}{\Delta x} Q_{i+1}^{n+1} + \alpha (Q_{i+1}^{n+1})^\beta = \frac{\Delta t}{\Delta x} Q_{i+1}^n + \alpha (Q_{i+1}^n)^\beta \quad \text{--- (i)}$$

$$f(Q_{i+1}^{n+1}) = \frac{\Delta t}{\Delta x} Q_{i+1}^{n+1} + \alpha (Q_{i+1}^{n+1})^\beta - c \quad \text{--- (ii)}$$

$$f'(Q_{i+1}^{n+1}) = \frac{\Delta t}{\Delta x} + \alpha \beta (Q_{i+1}^{n+1})^{\beta-1} \quad \text{--- (iii)}$$

$$(Q_{i+1}^{n+1})_{k+1} = (Q_{i+1}^{n+1})_k - \frac{f(Q_{i+1}^{n+1})}{f'(Q_{i+1}^{n+1})} \quad \text{--- (iv)}$$

$$c = \frac{1.5 \times 60 \times 60}{4650} \times 580 + 5 \times (420)^{0.6}$$

$$[c = 861.01]$$

Iteration table.

Trial no.	Trial value $(Q_{i+1}^{n+1})_k$	$f(Q_{i+1}^{n+1})_k$	$f'(Q_{i+1}^{n+1})_k$	$(Q_{i+1}^{n+1})_{k+1}$
1	500	-72.226	1.411	551.186
2	551.186	-0.2499	1.401	551.364
3	551.364	-2.772 × 10 ⁻⁶	1.401	551.364

$$\text{Thus, } Q_{i+1}^{n+1} = 551.364 \text{ m}^3/\text{s}$$